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Équations de réaction-diffusion dans des milieux hétérogènes non bornés

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à mes parents

à mes frères et soeurs

à mes grand-parents

Résumé

L'objet de cette thèse est l'étude de certains phénomènes de propagation de fronts pulsatoires pour des problèmes de réaction-advection-diffusion du modèle

$$\begin{cases} u_t = \nabla \cdot (A(z)\nabla u) \ + q(z) \cdot \nabla u + \ f(z,u), \ t \in \mathbb{R}, \ z \in \Omega, \\ \nu \cdot A \ \nabla u(t,z) = 0, \ t \in \mathbb{R}, \ z \in \partial\Omega, \end{cases}$$

où $\Omega \subseteq \mathbb{R}^N$ est un domaine périodique non borné. Les coefficients de l'équation et le domaine Ω seront périodiques par rapport aux variables d'espace. La thèse se compose de trois parties qui correspondent à trois articles soumis à des revues internationales avec comité de lecture. En fait, l'existence de fronts progressifs pulsatoires dépend fortement du type de la nonlinéarité. Si la nonlinéarité f est de type "KPP", il existe une vitesse minimale c^*. La première partie porte sur les comportements asymptotiques de la vitesse minimale c^* de propagation des ondes progressives dans le cas "KPP" (utilisant une formule variationnelle de c^* donnée par Berestycki, Hamel, et Nadirashvili en 2002). Dans la seconde partie, on donne des formules min − max et max − min pour les vitesses de propagation selon le type de la réaction. La troisième partie concerne la dépendance de la vitesse par rapport à la période spatiale dans un cadre plus général (concernant la diffusion et la nonlinéarité) que celui de la première partie, mais en dimension $N = 1$ seulement.

Mots clés: KPP, réaction-diffusion, front progressifs pulsatoires, vitesse de propagation, homogénéisation, biological invasion models, combustion, pulsating travelling fronts, minimal speed of propagation, homogenization, fragmentation.

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CHAPITRE 1

Introduction générale

L'objet de cette thèse est l'étude de certains phénomènes de propagation de fronts pulsatoires pour des problèmes de réaction-advection-diffusion non linéaires dans des milieux hétérogènes non bornés. Les coefficients de l'équation et le domaine Ω seront périodiques par rapport aux variables d'espace. En fait, l'existence de fronts progressifs pulsatoires dépend fortement du type de la nonlinéarité. La thèse se compose de trois parties qui correspondent à trois articles soumis à des revues internationales avec comité de lecture. Si la nonlinéarité f est de type "KPP", il existe une vitesse minimale c^*. La première partie porte sur les comportements asymptotiques de la vitesse minimale c^* de propagation des ondes progressives dans le cas "KPP". Grâce à une formule variationnelle pour cette vitesse (donnée par Berestycki, Hamel, et Nadirashvili [3]), nous traitons c^* comme une fonction du facteur de diffusion, du facteur de réaction, et du paramètre de périodicité afin d'étudier les variations et les comportements asymptotiques de c^* par rapport aux coefficients du problème. A la fin de cette partie, on applique les résultats obtenus pour résoudre un problème d'homogénéisation. Dans la seconde partie, on donne des formules $\min - \max$ et $\max - \min$ pour les vitesses de propagation selon le type de la réaction. La troisième partie concerne la dépendance de la vitesse par rapport à la période spatiale dans un cadre plus général (concernant la diffusion et la nonlinéarité) que celui de la première partie, mais en dimension $N = 1$ seulement.

Les équations de réaction-diffusion apparaissent naturellement dans la modélisation de lévolution de la température lors dune réaction de combustion, où la chaleur est créée par la réaction et diffuse selon la loi de la chaleur, et peut également être transportée dans le milieu (vent, aération), ce qui donne un terme dadvection supplémentaire. De même, létude de lévolution de la densité dune population animale dans un milieu dont les caractéristiques sont plus ou moins favorables à la survie et au développement de lespèce considérée se modélise par une équation de réaction-diffusion, où la diffusion traduit les mouvements de la population et le terme de réaction regroupe les naissances, les décès en prenant en compte linfluence du milieu (dépendance spatiale du terme de réaction), et les interactions avec le milieu ou dautres espèces.

Les premières analyses mathématiques des équations de réaction-diffusion ont été entreprises dans les années 1930, principalement létude de léquation unidimensionnelle

$$\partial_t u = \partial_{xx} u + f(u)$$
$$0 < u < 1.$$

En 1937, Fisher [9] et Kolmogorov, Petrovsky et Piskunov [21] ont étudié le modèle homogène

$$u_t - \Delta u = f(u) \text{ dans } \mathbb{R}^N, \tag{1.0.1}$$

avec une nonlinéarité f satisfaisant

$$\begin{cases} f(0) = f(1) = 0,\ f'(1) < 0,\ f'(0) > 0, \\ f > 0 \text{ dans } (0,1),\ f < 0 \text{ dans } (1,+\infty), \end{cases} \tag{1.0.2}$$

$$\forall s \in [0,1],\ f(s) \leq f'(0)s. \tag{1.0.3}$$

Comme archétypes de nonlinéarités “KPP”, on a $f(s) = s(1-s)$ et $f(s) = s(1-s^2)$.

Un front progressif qui se propage dans la direction d'un vecteur $-e \in \mathbb{R}^N$ ($|e| = 1$) est une solution u de (1.0.1) telle que

$$\begin{cases} \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ u(t,x) = \phi(x \cdot e + ct), \\ \phi(-\infty) = 0 \text{ et } \phi(+\infty) = 1, \end{cases} \tag{1.0.4}$$

La valeur c est appelée la vitesse du front dans la direction de $-e$. Suite à cette dernière définition, l'équation (1.0.1) est équivalente à

$$\begin{cases} -\phi'' + c\,\phi' = f(\phi) \quad \text{dans } \mathbb{R} \times \mathbb{R}^N, \\ \phi(-\infty) = 0 \text{ et } \phi(+\infty) = 1. \end{cases} \tag{1.0.5}$$

Kolmogorov, Petrovsky et Piskunov [21] ont montré qu'il existe une valeur $c^* = 2\sqrt{f'(0)}$ telle que les fronts progressifs (c,u) de l'équation (1.0.1) tels que $0 < u < 1$

existent si et seulement si $c \geq c^*(e) = 2\sqrt{f'(0)}$. De plus, pour chaque $c \geq 2\sqrt{f'(0)}$, le front u est unique à une translation près en t.

A partir de là, de très nombreux articles, parmi lesquels ceux de Aronson et Weinberger [1], Fife et McLeod [8], Johnson et Nachbar [18] ou encore Kanel [19] ont été consacrés à des problèmes liés à lexistence, à l'unicité et à la stabilité de telles ondes planes progressives, en considérant différents types de nonlinéarités et de domaines despace donnés par la modélisation des situations physiques, chimiques et biologiques étudiées. De nombreux travaux ont porté sur les systèmes en dimension 1 despace (voir par exemple [31]).

Lorsque le milieu nest plus considéré comme homogène, les coefficients de léquation, et en particulier le terme de réaction, dépendent également des variables despace ou de temps. On ne peut alors plus trouver de solutions en forme donde plane pour les nouvelles équations considérées, y compris en dimension 1. Cest pourquoi une nouvelle notion moins exigeante mais plus générale, celle donde progressive pulsatoire, cest-à-dire donde se propageant dans une direction donnée, au profil variable mais se reproduisant périodiquement, a été proposée, dans les milieux périodiques, par Shigesada, Kawasaki, et Teramoto [26] en 1985 et par Shigesada et Kawazaki [29] en 1997. Hudson et Zinner [16] ont montré en 1995 lexistence dondes progressives pulsatoires solutions de léquation en dimension 1. Concernant les fronts progressifs pour des équations de réaction-diffusion avec de coefficients dependant des variables de temps et d'espace, on peut voir Fréjacques [10], Nadin [22, 23], et Shen [27, 28].

1.1 Présentation du cadre hétérogène périodique et de quelques résultats connus

Dans les chapitres 2 et 3 on s'intéresse au modèle hétérogène suivant :

$$\begin{cases} u_t = \nabla \cdot (A(z)\nabla u) \ + q(z) \cdot \nabla u + \ f(z,u), \ t \in \mathbb{R}, \ z \in \Omega, \\ \nu \cdot A \ \nabla u(t,z) = 0, \ t \in \mathbb{R}, \ z \in \partial\Omega, \end{cases} \tag{1.1.1}$$

où $\nu(z)$ est la normale extérieure au point $z \in \partial\Omega$. La condition au bord $\nu \cdot A \nabla u(x,y) = 0$ est équivalente à

$$\sum_{1 \leq i,j \leq N} \nu_i(x,y) A_{ij}(x,y) \partial_{x_j} u(t,x,y) = 0.$$

On note que si $A = Id_{M_N(\mathbb{R})}$, la condition au bord est alors réduite à la condition de Neumann usuelle.

Dans la suite, on va détailler les hypothèses sur le domaine Ω, la matrice de diffusion A, le flot d'advection q et la réaction f qui apparaissent dans l'équation (1.1.1) :

Le domaine Ω est un ouvert connexe de $\mathbb{R}^N$ de classe C^3 qui satifait

$$\begin{cases} \exists\, R \geq 0,\ \exists\, d \geq 1 \in \mathbb{N},\ \forall z = (x,y) \in \Omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-d},\ |y| \leq R, \\ \exists\, L_1 > 0, \cdots, L_d > 0,\ \forall\, (k_1, \cdots, k_d) \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z},\ \Omega = \Omega + \sum_{k=1}^{d} k_i e_i, \end{cases} \tag{1.1.2}$$

où $(e_i)_{1\leq i\leq N}$ est la base canonique de $\mathbb{R}^N$. Comme $d \geq 1$, le domaine Ω est non borné.

Avant d'aller plus loin, on note que ce cadre généralise plusieurs types de configurations géométriques. Le cas où $\Omega = \mathbb{R}^N$ correspond à $d = N$ où $L_1, \cdots, L_N$ peuvent être des constantes positives quelconques. Dans ce cadre, on peut aussi considérer le cas de l'espace $\mathbb{R}^N$ avec un ensemble périodique de trous. Le cas $d = 1$ correspond à des domaines qui sont non bornés dans une seule direction. C'est-à- dire des cylindres infinis qui peuvent avoir une frontière oscillante, et qui peuvent contenir un ensemble périodique de trous.

Définitions 1.1.1 (Cellule de périodicité et champs L-périodiques) *Etant donnée un ensemble Ω satisfaisant (1.1.2), on définit l'ensemble*

$$C = \{(x,y) \in \Omega;\ x_1 \in (0, L_1), \cdots, x_d \in (0, L_d)\}$$

comme la cellule de périodicité de Ω. De plus, on dit que le champ $w : \Omega \mapsto \mathbb{R}^p$ est L-périodique par rapport à x si

$$\forall k = (k_1, \cdots, k_d) \in \prod_{i=1}^{d} L_i\mathbb{Z},$$
$$w(x_1 + k_1, \cdots, x_d + k_d\,, y) = w(x_1, \cdots, x_d, y) \quad p.p.\ dans\ \ \Omega.$$

Le terme de diffusion est une matrice symétrique $A = A(x,y) = (A_{ij}(x,y))_{1\leq i,j\leq N}$ de classe $C^{2,\delta}(\overline{\Omega})$ (avec $\delta > 0$) qui satisfait

$$\begin{cases} A \text{ est } L\text{-périodique par rapport à } x,\ \text{ et } \exists\, 0 < \alpha_1 \leq \alpha_2 \text{ tel que} \\ \forall (x,y) \in \Omega, \forall\, \xi \in \mathbb{R}^N, \alpha_1|\xi|^2 \leq \sum_{1\leq i,j\leq N} A_{ij}(x,y)\xi_i\xi_j \leq \alpha_2|\xi|^2. \end{cases} \tag{1.1.3}$$

L'advection $q = (q_1(x,y), \cdots, q_N(x,y))$ est un champ vectoriel de classe $C^{1,\delta}(\overline{\Omega})$

(avec $\delta > 0$) tel que

$$\begin{cases} q \text{ est } L\text{-périodique par rapport à } x, \\ \nabla \cdot q = 0 \quad \text{dans } \overline{\Omega}, \\ q \cdot \nu = 0 \quad \text{sur } \partial\Omega, \\ \forall\, 1 \leq i \leq d, \quad \displaystyle\int_C q_i \, dx\, dy = 0. \end{cases} \tag{1.1.4}$$

Finalement, le terme de réaction (ou la source nonlinéaire) est une fonction positive $f = f(x, y, u)$ définie dans $\overline{\Omega} \times \mathbb{R}$ telle que

$$\begin{cases} f \text{ est globalement Lipschitzienne dans } \overline{\Omega} \times \mathbb{R}, \\ \forall\,(x, y) \in \overline{\Omega},\ \forall\, s \in (-\infty, 0] \cup [1, +\infty),\ f(s, x, y) = 0, \\ \exists\, \rho \in (0, 1),\ \forall(x, y) \in \overline{\Omega},\ \forall\, 1 - \rho \leq s \leq s' \leq 1,\ f(x, y, s) \geq f(x, y, s'). \end{cases} \tag{1.1.5}$$

On suppose que

$$f \text{ est } L\text{-périodique par rapport à } x. \tag{1.1.6}$$

Par ailleurs, on suppose que la fonction f satisfait une des deux propriétés suivantes

$$\begin{cases} \exists\, \theta \in (0, 1),\ \forall(x, y) \in \overline{\Omega},\ \forall\, s \in [0, \theta],\ f(x, y, s) = 0, \\ \forall\, s \in (\theta, 1),\ \exists\,(x, y) \in \overline{\Omega} \text{ tel que } f(x, y, s) > 0, \end{cases} \tag{1.1.7}$$

ou

$$\begin{cases} \exists \delta > 0, \text{ la restriction de } f \text{ à } \overline{\Omega} \times [0, 1] \text{ est de classe } C^{1,\delta}, \\ \forall\, s \in (0, 1),\ \exists\,(x, y) \in \overline{\Omega} \text{ such that } f(x, y, s) > 0. \end{cases} \tag{1.1.8}$$

Définition 1.1.2 *Si la nonlinéarité f satisfait (1.1.5), (1.1.6) et (1.1.7), on dit que f est de type "combustion". La valeur θ est appelée la temperature d'ignition. Si f satisfait (1.1.5), (1.1.6) et (1.1.8), on dit alors que f est de type "ZFK" (pour Zeldovich - Frank Kamenetskii).*
Si f est une nonlinéairité "ZFK" satisfaisant les conditions

$$\zeta(x, y) = f'_u(x, y, 0) = \lim_{u \to 0^+} f(x, y, u)/u > 0, \ \textit{et} \tag{1.1.9}$$

$$\forall\,(x, y, s) \in \overline{\Omega} \times (0, 1), \quad 0 < f(x, y, s) \leq f'_u(x, y, 0) \times s, \tag{1.1.10}$$

on dit que f est de type "KPP" (pour Kolmogrov, Petrovsky, et Piskunov).

Dans le cadre hétérogène périodique décrit ci-dessus, on peut définir les fronts progressifs pulsatoires :

Définition 1.1.3 *Soient* $e = (e^1, \cdots, e^d) \in \mathbb{R}^d$ *un vecteur unitaire* ($|e| = 1$) *et* $\tilde{e} = (e^1, \cdots, e^d, 0, \cdots, 0) \in \mathbb{R}^N$. *On dit que la fonction* $u = u(t, x, y)$ *est un front progressif pulsatoire qui se propage dans la direction de* $-e$ *avec une vitesse* $c \neq 0$ *si* u *est une solution classique de*

$$\begin{cases} u_t = \nabla \cdot (A(x,y)\nabla u) + q(x,y) \cdot \nabla u + f(x,y,u), \ t \in \mathbb{R}, \ (x,y) \in \Omega, \\ \\ \nu \cdot A \ \nabla u(t,x,y) = 0, \ t \in \mathbb{R}, \ (x,y) \in \partial\Omega, \\ \forall\, k \in \prod_{i=1}^{d} L_i\mathbb{Z}, \ \forall\, (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad u(t + \dfrac{k \cdot e}{c}, x, y) = u(t, x+k, y), \\ \lim\limits_{x.e \to -\infty} u(t,x,y) = 0, \ and \ \lim\limits_{x.e \to +\infty} u(t,x,y) = 1, \\ 0 \leq u \leq 1, \end{cases} \tag{1.1.11}$$

où les limites ci-dessus sont locales en t *et uniformes en* y *et toutes les directions de* $\mathbb{R}^d$ *qui sont orthogonales à* e.

Plusieurs articles et travaux ont été consacrés à l'analyse des phénomènes de propagation pour des équations de réaction-diffusion ou d'autres types d'équations hétérogènes périodiques. Comme résultats d'existence de fronts progressifs (c, u) dans des milieux hétérogènes nonbornés et périodiques on peut voir Namah, Roquejoffre [24], Papanicolaou, Xin [25], J. Xin [32], J. Xin [33], et X. Xin [34] dans le cas unidimensionnelle ou dans le cas $\Omega = \mathbb{R}^N$. Pour l'existence dans des problèmes similaires, on peut voir Brauner, Fife, Namah [4] et Heinze [13]. Dans le cadre hétérogène périodique décrit ci-dessus, Berestycki et Hamel [2] ont traité la question d'existence de fronts progressifs pulsatoires selon la non linéarité f du modèle (1.1.11) :

Théorème 1.1.4 (Berestycki, Hamel [2]) *Si la nonlinéarité* f *est de type "Combustion", alors pour toute direction* e *de* $\mathbb{R}^d$, *il existe une solution classique* (c, u) *de (1.1.11). De plus, la vitesse* c *est positive et unique, et la fonction* $u = u(t, x, y)$ *est croissante en* t *et elle est unique à une translation près en* t. *Précisément, si* (c^1, u^1) *et* (c^2, u^2) *sont deux solutions classiques de (1.1.11), alors* $c^1 = c^2$ *et il existe* $h \in \mathbb{R}$ *telle que* $u^1(t, x, y) = u^2(t + h, x, y)$ *pour tout* $(t, x, y) \in \mathbb{R} \times \overline{\Omega}$.

Théorème 1.1.5 (Berestycki, Hamel [2]) *Si* f *est de type "ZFK", alors il existe* $c^*_{\Omega,A,q,f}(e) > 0$ *telle que problème (1.1.11) n'a pas une solution* (c, u), *avec* $u_t > 0$ *dans* $\mathbb{R} \times \overline{\Omega}$, *si* $c < c^*_{\Omega,A,q,f}(e)$. *Par contre, pour tout* $c \geq c^*_{\Omega,A,q,f}(e)$, *il existe une solution* (c, u) *avec* $u_t > 0$. *La valeur* $c^*_{\Omega,A,q,f}(e)$ *est appelée la vitesse minimale de propagation du problème (1.1.11).*

Très recemment, Hamel et Roques [12] ont montré l'unicité de fronts pulsatoires progressifs, à une translation près en t, pour toute $c \geq c^*_{\Omega,A,q,f}(e)$ lorsque la nonlinéarité f est de type "KPP."

Le théorème 1.1.5 s'applique en particulièr dans le cas "KPP". De plus, Berestycki, Hamel et Nadirashvili [3] ont montré une formule variationnelle pour la vitesse minimale $c^*_{\Omega,A,q,f}(e)$ du modèle (1.1.11) avec une nonlinéarité de type "KPP" :

Théorème 1.1.6 (Berestycki, Hamel, Nadirashvili [3]) *Lorsque la nonlinéarité f est de type "KPP", la vitesse minimale $c^*_{\Omega,A,q,f}(e)$ de propagation dans la direction de $-e$ du problème (1.1.11) est donnée par*

$$c^*_{\Omega,A,q,f}(e) \;=\; \min_{\lambda>0} \frac{k_{\Omega,e,A,q,\zeta}(\lambda)}{\lambda}, \tag{1.1.12}$$

où $k_{\Omega,e,A,q,\zeta}(\lambda)$ est la valeur propre principale de l'opérateur $L_{\Omega,e,A,q,\zeta,\lambda}$ défini par

$$\begin{array}{rcl} L_{\Omega,e,A,q,\zeta,\lambda}\psi & := & \nabla\cdot(A\nabla\psi) + 2\lambda\tilde{e}\cdot A\nabla\psi + q\cdot\nabla\psi \\ & & +[\lambda^2\tilde{e}A\tilde{e} + \lambda\nabla\cdot(A\tilde{e}) + \lambda q\cdot\tilde{e} + \zeta]\psi \;\; dans\ \Omega, \end{array} \tag{1.1.13}$$

sur l'ensemble

$$\begin{array}{rl} E_\lambda = & \big\{\psi = \psi(x,y) \in C^2(\overline{\Omega}), \psi\ L\text{-périodique par rapport à } x \text{ et} \\ & \nu\cdot A\nabla\psi(x,y) \;=\; -\lambda\,\nu\cdot A\tilde{e}\,\psi(x,y) \text{ sur } \partial\Omega\big\}\,. \end{array}$$

Dans le cadre homogène où $\Omega = \mathbb{R}^N$, $f = f(u)$, $q = 0$ et $A = Id$, on note que (1.1.12) implique la formule KPP $c^* = 2\sqrt{f'(0)}$ (voir [21]). En fait, la fonction propre principale, qui est positive et unique à multiplication par une constante positive près, sera $\psi =$ constante ($\zeta = f'(0)$ dans ce cas). Donc, pour tout $\lambda > 0$, on aura $k(\lambda) = \lambda^2 + f'(0)$ et alors $\displaystyle\min_{\lambda>0}\frac{k(\lambda)}{\lambda} = 2\sqrt{f'(0)}$. La formule (1.1.12) est très utile dans l'analyse des comportements asymptotiques et des variations de la vitesse minimale c^* par rapport aux coefficients de diffusion, de réaction et d'advection et par rapport aux paramètres de périodicité et la géométrie du domaine Ω dans le modèle (1.1.11). Utilisant cette formule, plusieurs articles ont étudié de tels problèmes (voir par exemple Berestycki, Hamel, Nadirashvili [3], El Smaily [5], El Smaily [7], Heinze [14], Ryzhik, Zlatoš [30] et Zlatoš [35]).

1.2 Résultats principaux

Dans cette thèse, nous traitons le problème de réaction-advection-diffusion dans le cadre hétérogène-périodique de la section 1.1.

Dans toute cette section, $e \in \mathbb{R}^d$ est un vecteur tel que $|e| = 1$ et $\tilde{e} = (e, 0, \cdots, 0) \in \mathbb{R}^N$.

Le chapitre 2 correspond à l'article El Smaily [5]. Il est consacré à l'analyse asymptotique des variations de la vitesse minimale de propagation par rapport aux coefficients de diffusion, de réaction et d'advection et par rapport aux paramètres de périodicité dans le cas "KPP".

Concernant la vitesse minimale en présence d'une grande diffusion MA (avec $M > 0$), on a montré le théorème suivant

Théorème 1.2.1 (El Smaily [5]) *Soient f de type "KPP" (voir Définition 1.1.2), Ω, A et q satisfaisant (1.1.2), (1.1.3) et (1.1.4) respectivement. Nous supposons aussi que $\nabla . A\tilde{e} \equiv 0$ dans Ω et $\nu \cdot A\tilde{e} = 0$ sur $\partial\Omega$. Pour tous $M > 0$ et $0 \leq \gamma \leq 1/2$, considérons le problème*

$$\begin{cases} u_t = M\,\nabla \cdot (A(x,y)\nabla u) + M^{\gamma}\, q(x,y) \cdot \nabla u + f(x,y,u),\ t \in \mathbb{R},\ (x,y) \in \Omega, \\ \nu \cdot A\ \nabla u(t,x,y) = 0,\ t\ \in\ \mathbb{R},\ (x,y)\ \in\ \partial\Omega, \end{cases} \tag{1.2.1}$$

dont la diffusion est la matrice MA. Alors,

$$\lim_{M \to +\infty} \frac{c^*_{\Omega, MA, M^{\gamma} q, f}(e)}{\sqrt{M}} = 2\sqrt{⨍_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\ \sqrt{⨍_C \zeta(x,y)dx\,dy},$$

où

$$⨍_C \tilde{e}A\tilde{e}(x,y)dx\,dy = \frac{1}{|C|}\int_C \tilde{e}A\tilde{e}(x,y)dx\,dy,$$

et $|C|$ est la mesure de Lebesgue de la cellule de périodicité C de Ω.

Grâce à ce théorème, on a trouvé la limite par homogénéisation de la vitesse minimale lorsque la cellule de péiodicité $C^{\varepsilon} = \varepsilon C$ devient très petite. Pour tout $\varepsilon > 0$, on définit A_{ε}, q_{ε}, et f_{ε} comme

$$\begin{gathered} \forall (x,y) \in \Omega, \quad A_{\varepsilon}(x,y) = A\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), q_{\varepsilon}(x,y) = q\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), \\ \text{et } f_{\varepsilon}(x,y,u) = f\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, u\right). \end{gathered} \tag{1.2.2}$$

Théorème 1.2.2 (El Smaily [5]) *Soit* Ω *un domaine satisfaisant (1.1.2) avec une cellule de périodicité* C. *Pour tout* $\varepsilon > 0$, *on pose* $\Omega^\varepsilon = \varepsilon\Omega$. *Sous les mêmes hypothèses que dans le Théorème 1.2.1, on considère le problème*

$$\begin{cases} u_t^\varepsilon(t,x,y) = \nabla\cdot(A_\varepsilon \nabla u^\varepsilon)(t,x,y) + q_\varepsilon\cdot\nabla u^\varepsilon + f_\varepsilon(x,y,u^\varepsilon), \ \ dans \ \ \mathbb{R}\times\Omega^\varepsilon, \\ \nu^\varepsilon\cdot A_\varepsilon\,\nabla u^\varepsilon(t,x,y) = 0, \quad t\in\mathbb{R},\ (x,y)\in\partial\Omega^\varepsilon, \end{cases} \tag{1.2.3}$$

Alors,

$$\lim_{\varepsilon\to 0^+} c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,\,f_\varepsilon}(e) = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\sqrt{\fint_C \zeta(x,y)dx\,dy}. \tag{1.2.4}$$

Concernant le problème d'homogénéisation lorsque $\Omega = \mathbb{R}^N$,

$$\begin{aligned} u_t(t,x,y) &= \nabla\cdot(A_L\nabla u)(t,x,y) + q_L\cdot\nabla u(t,x,y) + f_L(x,y,u),, \\ &= \nabla\cdot(A(\frac{x}{L},\frac{y}{L})\nabla u)(t,x,y) + q(\frac{x}{L},\frac{y}{L})\cdot\nabla u(t,x,y) + f(\frac{x}{L},\frac{y}{L},u), \end{aligned} \tag{1.2.5}$$

on a obtenu le résultat suivant :

Lorsque f *est de type "KPP",* A *et* q *satisfont (1.1.3) et (1.1.4) respectivement. Si* $\nabla.A\tilde{e}\equiv 0$ *dans* $\mathbb{R}^N$, *alors*

$$\lim_{L\to 0^+} c^*_{\mathbb{R}^N,\,A_L,\,q_L,\,f_L}(e) = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\sqrt{\fint_C \zeta(x,y)dx\,dy}. \tag{1.2.6}$$

Le théorème suivant donne un équivalent de la vitesse minimale avec une petite réaction Bf $(B>0)$:

Théorème 1.2.3 (El Smaily [5]) *Soient* $B>0$, $\gamma\geq 1/2$ *et* f *une nonlinéarité de type "KPP". Supposons que* Ω, A *et* q *satisfont (1.1.2), (1.1.3) et (1.1.4) respectivement,* $\nabla.A\tilde{e}\equiv 0$ *dans* Ω *et* $\nu\cdot A\tilde{e}=0$ *sur* $\partial\Omega$. *Alors,*

$$\lim_{B\to 0^+}\frac{c^*_{\Omega,A,B^\gamma q,Bf}(e)}{\sqrt{B}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\sqrt{\fint_C \zeta(x,y)dx\,dy}.$$

Pour le comportement asymptotique de la vitesse minimale de propagation en présence d'une petite diffusion εA $(\varepsilon>0)$, nous considérons un cadre moins général que celui au Théorème 1.2.1 que nous décrivons ci-dessous.

Dans ce cadre, nous choisissons $\Omega = \mathbb{R}\times\omega \subseteq \mathbb{R}^N$, avec $\omega\subseteq\mathbb{R}^d\times\mathbb{R}^{N-d-1}$ $(d\geq 0)$. Si $d=0$ alors ω est un domaine borné et connexe de classe C^3 dans $\mathbb{R}^{N-1}$. Par contre, si

$1 \leq d \leq N-1$, alors ω est un domaine $(L_1, \ldots, L_d)$-périodique dans $\mathbb{R}^{N-1}$ qui satisfait (1.1.2). Donc, Ω est un domaine de $\mathbb{R}^N$ qui est $(l, L_1, \ldots, L_d)-$périodique (pour tout $l > 0$) satisfaisant (1.1.2). Un élément de $\Omega = \mathbb{R} \times \omega$ aura alors la forme $z = (x, y)$ avec $x \in \mathbb{R}$ et $y \in \omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-1-d}$.

Concernant la nonlinéarité $f = f(x, y, u)$, elle est de type “KPP” telle que

$$\begin{cases} f \geq 0, \text{ et est de classe } C^{1,\delta}(\mathbb{R} \times \overline{\omega} \times [0,1]), \\ f \text{ est } (l, L_1, \ldots, L_d)\text{-périodique par rapport à } (x, y_1, \ldots, y_d), \text{ lorsque } d \geq 1, \\ f \text{ est } l\text{-périodique par rapport à } x, \text{ lorsque } d = 0, \end{cases} \quad (1.2.7)$$

et

$$\begin{cases} \forall (x,y) \in \overline{\Omega},\ f'_u(x,y,0) \text{ ne dépend que de } y. \text{ Posons } \zeta(y) = f'_u(x,y,0). \\ \forall\, (x,y) \in \overline{\Omega} = \mathbb{R} \times \overline{\omega}, \quad f'_u(x,y,0) = \zeta(y) > 0, \\ \forall\, (x,y,s) \in \overline{\Omega} \times (0,1), \quad 0 < f(x,y,s) \leq \zeta(y)\, s. \end{cases} \quad (1.2.8)$$

Remarquons qu'il est possible de trouver une nonlinéarité f telle que $f'_u(x, y, u)$ ne dépend que de y tandis que $f(x, y, u)$ dépend de x, y et u.

Enfin, la diffusion est une matrice symétrique $A(x, y) = A(y) = (A_{ij}(y))_{1 \leq i,j \leq N}$ de classe $C^{2,\delta}(\overline{\Omega})$ $(\delta > 0)$ qui ne dépend que de y et qui satisfait

$$\begin{cases} A \text{ est } (L_1, \ldots, L_d)\text{-périodique par rapport à } (y_1, \ldots, y_d), \\ \exists\, 0 < \alpha_1 \leq \alpha_2,\ \forall\, y \in \omega, \forall\, \xi \in \mathbb{R}^N, \alpha_1 |\xi|^2 \leq \displaystyle\sum_{1 \leq i,j \leq N} A_{ij}(y) \xi_i \xi_j \leq \alpha_2 |\xi|^2. \end{cases} \quad (1.2.9)$$

Théorème 1.2.4 (El Smaily [5]) *Soit $e = (1, 0, \ldots, 0) \in \mathbb{R}^N$ et $\varepsilon > 0$. Soient $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$ un domaine sous la forme décrite ci-dessus, f une nonlinéarité de type “KPP” satisfaisant (1.2.7), (1.2.8), et A une matrice satisfaisant (1.2.9). Considérons l'équation de réaction-diffusion*

$$\begin{cases} u_t(t,x,y) = \varepsilon \nabla \cdot (A(y) \nabla u)(t,x,y) + f(x,y,u), \ \textit{for} \ (t,x,y) \in \mathbb{R} \times \Omega, \\ \nu \cdot A \nabla u = 0 \quad \textit{on} \quad \mathbb{R} \times \mathbb{R} \times \partial\omega. \end{cases} \quad (1.2.10)$$

De plus, supposons que A et f satisfont une des deux alternatives suivantes

$$\begin{cases} \exists\, \alpha > 0,\ \forall y \in \omega,\ A(y) e = \alpha e, \\ f'_u(x,y,0) = \zeta(y), \textit{ pour tout } (x,y) \in \overline{\Omega}, \end{cases} \quad (1.2.11)$$

ou

$$\begin{cases} f'_u(x,y,0) = \zeta \ \text{ est constante,} \\ \forall y \in \omega,\ A(y)e = \alpha(y)e,\ \text{où} \\ y \mapsto \alpha(y) \text{ est une fonction positive et } (L_1,\ldots,L_d)-\text{périodique dans } \overline{\omega}. \end{cases} \tag{1.2.12}$$

Alors,

$$\lim_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} = 2\sqrt{\max_{\overline{w}} \zeta}\sqrt{\max_{\overline{w}} eAe}. \tag{1.2.13}$$

Remarque 1.2.5 (En présence d'une advection) *Sous les mêmes hypothèses du Théorème 1.2.4, prenons* $q = (q_1(y), 0, \ldots, 0)$ *(*$y \in \overline{\omega}$*) où* $q \not\equiv 0$ *dans* $\mathbb{R}\times\overline{\omega}$ *et* $\int_\omega q_1 = 0$. *Considérons l'équation de réaction-advection-diffusion*

$$\begin{cases} u_t = \varepsilon\nabla\cdot(A(y)\nabla u) + q_1(y)\,\partial_x u(t,x,y) + f(x,y,u),\ t\in\mathbb{R},\ (x,y)\in\Omega, \\ \nu\cdot A\,\nabla u(t,x,y) = 0, \quad t\in\mathbb{R}, \quad (x,y)\in\partial\Omega, \end{cases} \tag{1.2.14}$$

Utilisant les mêmes techniques que dans la preuve du Théoreme 1.2.4, on peut montrer que

$$\lim_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,q,f}(e) = \max_{y\in\overline{\omega}}\,(-\,q_1(y)) = \max_{\overline{\omega}}(-\,q.e). \tag{1.2.15}$$

Le Théorème 1.2.4 implique le théorème suivant :

Théorème 1.2.6 (El Smaily [5]) *Soit* $e = (1,0,\ldots,0) \in \mathbb{R}^N$, $\omega = \mathbb{R}^{N-1}$, $d = N-1$, *et* $l = L_1 = \ldots = L_{N-1} = 1$. *C'est-à-dire les coefficients de l'équation sont* $(1,1,\ldots,1)-$*périodiques par rapport à* y. *Un élément* $z\in\mathbb{R}^N$ *aura la forme* $z = (x,y)\in\mathbb{R}\times\mathbb{R}^{N-1}$. *Supposons que* $f(x,y,u)$ *et* $A = A(y)$ *satisfont (1.2.7), (1.2.8) et (1.2.9) avec une des deux alternatives (1.2.11) ou (1.2.12). Pour tous* $L > 0$, *et* $(x,y)\in\mathbb{R}^N$, *notons* $A_L(y) = A(\frac{y}{L})$ *et* $f_L(x,y,u) = f(\frac{x}{L},\frac{y}{L},u)$. *Considérons le problème de réaction-diffusion*

$$\begin{aligned} u_t(t,x,y) &= \nabla\cdot(A_L\nabla u)(t,x,y) + f_L(x,y,u),\ (t,x,y)\in\mathbb{R}\times\mathbb{R}^N \\ &= \nabla\cdot(A(\frac{y}{L})\nabla u)(t,x,y) + f(\frac{x}{L},\frac{y}{L},u),\ (t,x,y)\in\mathbb{R}\times\mathbb{R}^N, \end{aligned} \tag{1.2.16}$$

sont $(L,\ldots,L)$*-périodiques par rapport à* $(x,y)\in\mathbb{R}^N$. *Alors,*

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N,A_L,0,f_L}(e) = 2\sqrt{\max_{y\in\mathbb{R}^{N-1}} \zeta(y)}\sqrt{\max_{y\in\mathbb{R}^{N-1}} e.Ae(y)}. \tag{1.2.17}$$

Remontant au résultat (1.2.6), on remarque que ce théorème donne la limite de $c^*_{\mathbb{R}^N, A_L, q_L, f_L}(e)$ quand $L \to 0^+$ sous l'hypothèse $\nabla.A\tilde{e} \equiv 0$ dans $\mathbb{R}^N$. Donc, pour $N = 1$, ce théorème s'applique seulement lorsque la diffusion $x \mapsto a(x)$ est constante sur $\mathbb{R}$. Dans le chapitre 4 qui correspond à l'article [7], nous étudions, sans l'hypothèse de diffusion constante, le modèle

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(a_L(x)\dfrac{\partial u}{\partial x}\right) + f_L(x,u), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \\ \forall\, k \in \mathbb{Z},\ \forall\,(t,x) \in \mathbb{R} \times \mathbb{R}, \quad u(t + \dfrac{kL}{c}, x) = u(t, x+kL), \\ \lim\limits_{x \to -\infty} u(t,x) = 0 \ \text{ et } \ \lim\limits_{x \to +\infty} u(t,x) = 1, \end{cases} \tag{1.2.18}$$

les limites ci-dessus étant locales par rapport à t.

Le terme de diffusion a_L satisfait

$$a_L(x) = a(x/L),$$

a est une fonction 1-périodique de classe $C^{2,\delta}(\mathbb{R})$ (avec $\delta > 0$) qui satisfait

$$\exists\, 0 < \alpha_1 < \alpha_2,\ \forall\, x \in \mathbb{R},\ \alpha_1 \le a(x) \le \alpha_2. \tag{1.2.19}$$

Le terme de réaction est la fonction $f_L(x,\cdot) = f(x/L,\cdot)$, où $f := f(x,s)\ : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ est 1-périodique par rapport à x, de classe $C^{1,\delta}$ en (x,s) et C^2 en s. Donc, a_L et f_L sont L-périodiques par rapport à x. De plus, on suppose que

$$\begin{cases} \forall\, x \in \mathbb{R}, \quad f(x,0) = 0, \\ \exists\, M \ge 0,\ \forall\, s \ge M,\ \forall\, x \in \mathbb{R}, \quad f(x,s) \le 0, \\ \forall\, x \in \mathbb{R}, \quad s \mapsto f(x,s)/s \text{ est décroissante par rapport à } s > 0. \end{cases} \tag{1.2.20}$$

On pose

$$\mu(x) := \lim_{s \to 0^+} f(x,s)/s,$$

et

$$\mu_L(x) := \lim_{s \to 0^+} f_L(x,s)/s = \mu\left(\frac{x}{L}\right).$$

Le taux de croissance μ peut être positif dans quelques régions (régions favorables) ou négatif dans d'autres (régions défavorables). De plus, nous supposons que

$$\int_0^1 \mu(x)dx > 0. \tag{1.2.21}$$

Dans le cadre décrit ci-dessus, nous avons montré les théorèmes suivants (voir Chapitre 4) qui généralisent des résultats donnés par Kinezaki, Kawasaki, Takasu, et Shigesada [20] :

Théorème 1.2.7 (El Smaily, Hamel, Roques [7]) *Dans le cadre décrit ci-dessus, supposons que* $c^*_{\mathbb{R},\, a_L,\, 0,\, f_L}$ *est la vitesse minimale du modèle (1.2.18). Alors,*

$$\lim_{L\to 0^+} c^*_{\mathbb{R},\, a_L,\, 0,\, f_L} = 2\sqrt{<a>_H\ <\mu>_A}, \tag{1.2.22}$$

où

$$<\mu>_A = \int_0^1 \mu(x)dx \quad et \quad <a>_H = \left(\int_0^1 (a(x))^{-1}dx\right)^{-1} = <a^{-1}>_A^{-1}$$

sont, respectivement, la moyenne arithmétique de μ *et la moyenne harmonique de* a *sur l'intervalle* $[0,1]$.

Concernant les variations de $L \mapsto c^*_{\mathbb{R},\, a_L,\, 0,\, f_L}$, nous avons obtenu le résultat suivant :

Théorème 1.2.8 (El Smaily, Hamel, Roques [7]) *Sous les hypothèses du Théorème 1.2.7, la fonction* $L \mapsto c^*_{\mathbb{R},\, a_L,\, 0,\, f_L}$ *est de classe* C^∞ *dans un intervalle* $(0, L_0)$ *pour certain* $L_0 > 0$. *De plus,*

$$\lim_{L\to 0^+} \frac{dc^*_{\mathbb{R},\, a_L,\, 0,\, f_L}}{dL} = 0 \tag{1.2.23}$$

et

$$\lim_{L\to 0^+} \frac{d^2c^*_{\mathbb{R},\, a_L,\, 0,\, f_L}}{dL^2} = \gamma \geq 0. \tag{1.2.24}$$

Enfin, $\gamma > 0$ *si et seulement si la fonction*

$$\frac{\mu}{<\mu>_A} + \frac{<a>_H}{a}$$

n'est pas identiquement égale à 2.

Corollaire 1.2.9 *Avec les notations de Théorème 1.2.8, si* μ *est constante et* a *n'est pas constante, ou si* a *est constante et* μ *n'est pas constante, alors* $\gamma > 0$ *et donc* $L \mapsto c^*_{\mathbb{R},\, a_L,\, 0,\, f_L}$ *est strictement croissainte sur un intervalle* $]0, L_0]$ *pour certain* $L_0 > 0$.

Dans le chapitre 3, on donne des formules min-max et max-min pour les vitesse d'ondes progressives dans le cadre hétérogène périodique. Ces formules généralisent celles données par Hamel [11], Heinze, Papanicolaou, Stevens [15], et A.I Volpert, V.A Volpert, V.A Volpert [31].

Notations 1.2.10 *Notons*

$$\begin{aligned} E = \big\{ & \varphi = \varphi(s,x,y),\ \varphi \text{ est de classe } C^{1,\mu}(\mathbb{R}\times\overline{\Omega}) \ \text{ pour tout } \ \mu\in[0,1), \\ & F[\varphi]\in C(\mathbb{R}\times\overline{\Omega}),\ \varphi \text{ est } L-\text{périodique par rapport à } x,\ \varphi_s(s,x,y)>0 \\ & \text{dans } \mathbb{R}\times\overline{\Omega},\ \varphi(-\infty,.,.)=0,\ \varphi(+\infty,.,.)=1 \text{ uniformément dans } \overline{\Omega}, \\ & \text{et } \ \nu\cdot A(\nabla_{x,y}\varphi+\tilde{e}\varphi_s)=0 \text{ sur } \mathbb{R}\times\partial\Omega \big\}. \end{aligned}$$

Pour tout $\phi\in E$, nous notons $R\varphi\in C(\mathbb{R}\times\overline{\Omega})$ la fonction telle que

$$\begin{aligned} &\forall\ (s,x,y)\in\mathbb{R}\times\overline{\Omega}, \\ &\qquad R\,\varphi(s,x,y) = \ \frac{F[\varphi](s,x,y) + \ q\cdot\nabla_{x,y}\varphi(s,x,y) + f(x,y,\varphi)}{\partial_s\varphi(s,x,y)} + q(x,y)\cdot\tilde{e}. \end{aligned}$$

Théorème 1.2.11 (El Smaily [6]) *Soient e un vecteur de $\mathbb{R}^d$ tel que $|e|=1$, Ω un domaine satisfaisant (1.1.2) et f une nonlinéarité satisfaisant (1.1.5) et (1.1.6). De plus, nous supposons que A et q satisfont (1.1.3) and (1.1.4) respectivement. Si f est de type "combustion" satisfaisant (1.1.7), alors la vitesse unique $c(e)$ qui correspond au problème (1.1.11) est donnée par*

$$\begin{aligned} c(e) \ &= \ \min_{\varphi\in E}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y), && (1.2.25) \\ &= \ \max_{\varphi\in E}\ \inf_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y). && (1.2.26) \end{aligned}$$

(voir Notations 1.2.10).

De plus, le min *dans (1.2.25) et le* max *dans (1.2.26) sont atteints par, et seulement par, la fonction $\phi(s,x,y) = u\left(\frac{s-x\cdot e}{c(e)},x,y\right)$ et ses translations $\phi(s+\tau,x,y)$ pour tout $\tau\in\mathbb{R}$, où $(c(e),u)$ est la solution de (1.1.11) qui se propage avec la vitesse $c(e)$, dont l'existence et l'unicité (u est unique à une translation en t) ont été montrés dans le Théorème 1.1.4 de Berestycki et Hamel [2].*

Théorème 1.2.12 (El Smaily [6]) *Utilisons les notations 1.2.10 et supposons que la nonlinéarité f est de type "ZFK" (satisfaisant (1.1.8)). Alors, la vitesse minimale de propagation $c^*_{\Omega,A,q,f}(e)$ dans la direction de $-e$ est donnée par*

$$c^*_{\Omega,A,q,f}(e) = \min_{\varphi\in E}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y). \qquad (1.2.27)$$

En particulière, le Théorème 3.1.9 s'applique lorseque la non linéarité f est de type "KPP".

Remarque 1.2.13 *Dans le Théorème 1.2.11,* le min *et le* max *sont atteints par, et* *seulement par, le front pulsatoire $\phi(s,x,y)$ et ses translations $\phi(s+\tau,x,y)$ pour tout $\tau\in\mathbb{R}$. Dans le Théorème 3.1.9, le* min *est réalisé par le front pulsatoire $\phi^*(s,x,y)$ qui se propage avec la vitesse $c^*(e)$ et toute ses translations $\phi^*(s+\tau,x,y)$. Actuellement, si le front pulsatoire ϕ^* est unique à translation près, alors ϕ^* et ses translations en s sont les seules minimiseurs dans la formule (1.2.27). Nous rappelons que l'unicité de ϕ^* était récemment montrée par Hamel et Roques [12] dans le cas "KPP", mais elle n'est pas encore montrée dans le cas "ZFK" général.*

1.3 Problèmes ouverts, perspectives

Cette thèse propose une piste pour étudier plusieurs questions ouvertes concernant les équations de réaction-advection-diffusion :

I. Quelques questions concernant les analyses asymptotiques des vitesses de propagation dans un cadre hétérogène périodique :

1. Dans l'article El Smaily [5], nous avons étudié le comportement asymptotique de $M\mapsto c^*_{\Omega,MA,\,M^\gamma q,\,f}(e)$ lorsque $M\to+\infty$ sous l'hypothèse $0\le\gamma\le 1/2$. Nous avons trouvé que l'advection n'a pas d'influence sur la limite, lorsque $M\to+\infty$, de

$$\frac{c^*_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}}.$$

Il sera intéressant d'étudier le comportement asymptotique de

$$M\mapsto c^*_{\Omega,MA,M^\gamma q,f}(e)$$

dans le cas où $\gamma>1/2$ et f est du type "KPP."

2. Nous notons que la formule variationnelle de Berestycki, Hamel et Nadirashvili (1.1.12) a été l'outil essentiel pour étudier les asymptotiques de la vitesse minimale par rapport aux coefficients de l'équation de réaction-advection-diffusion dans le cas "KPP". Cette formule n'est plus valable lorsque f est de type "ZFK". Dans un cadre hétérogène et périodique comme celui de la section 1.1, si la nonlinéarité f est de type "ZFK" qui satisfait la condition

$$\forall(x,y)\in\overline{\Omega},\quad f'_u(x,y,0)=\lim_{u\to0^+}\frac{f(x,y,u)}{u}>0,$$

alors il existe deux nonlinéarités g et h de type "KPP" telles que $g\le f\le h$ dans $\mathbb{R}\times\Omega$.

La formule min-max (1.2.27) donée par El Smaily [6] implique que

$$c^*_{\Omega,A,\,q,\,g}(e) \leq c^*_{\Omega,A,q,f}(e) \leq c^*_{\Omega,A,q,h}(e),$$

$$\forall M > 0, \forall \gamma \in \mathbb{R},\; c^*_{\Omega,MA,M^\gamma q,g}(e) \leq c^*_{\Omega,MA,M^\gamma q,f}(e) \leq c^*_{\Omega,MA,M^\gamma q,h}(e),$$

et

$$\forall B > 0,\; c^*_{\Omega,A,B^\gamma q,Bg}(e) \leq c^*_{\Omega,A,B^\gamma q,Bf}(e) \leq c^*_{\Omega,A,B^\gamma q,Bh}(e).$$

Grâce à ces estimations pour les vitesses minimales et aux limites données par El Smaily [5], il sera ensuite important d'étudier les comportements asymptotiques de la vitesse minimale dans un cadre hétérogène périodique lorsque la nonlinéarité f est de type "ZFK" satisfaisant $f'_u(x,y,0) > 0$ dans $\overline{\Omega}$.

II. Problèmes d'optimisation : la formule min-max (1.2.27) de El Smaily [6] implique que l'application $\Phi : e \in \mathbb{S}^{d-1} \mapsto c^*_{\Omega,A,q,f}(e)$, $\mathbb{S}^{d-1}$ étant la sphère unitaire de $\mathbb{R}^d$, est continue lorsque f est de type "ZFK" et en particulier lorsque f est de type "KPP". En outre, dans le cas "KPP", la formule (1.1.12) de Berestycki, Hamel, Nadirashvili [3] implique aussi la continuité de l'application Φ. D'autre part, la formule (1.2.25) de El Smaily [6] implique que l'application $\Psi : e \in \mathbb{S}^{d-1} \mapsto c_{\Omega,A,q,f(e)}$ est continue lorsque f est de type "combustion". Suite à la continuité de Φ et Ψ sur le compact $\mathbb{S}^{d-1}$ de $\mathbb{R}^N$, il sera intéressant de trouver les directions $e \in \mathbb{S}^{d-1}$ où Φ et Ψ atteignent leur maximum et minimum, et de les caractériser en fonction des coefficients de diffusion, d'advection et de réaction, et en fonction de la géométrie du domaine Ω lorseque $\Omega \neq \mathbb{R}^N$.

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CHAPTER 2

Pulsating travelling fronts: Asymptotics and homogenization regimes

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Abstract. This paper is concerned with some nonlinear propagation phenomena for reaction-advection-diffusion equations with Kolmogrov-Petrovsky-Piskunov (KPP) type nonlinearities in general periodic domains or in infinite cylinders with oscillating boundaries. Having a variational formula for the minimal speed of propagation involving eigenvalue problems (proved in Berestycki, Hamel and Nadirashvili [3]), we consider the minimal speed of propagation as a function of diffusion factors, reaction factors and periodicity parameters. There we study the limits, the asymptotic behaviors and the variations of the considered functions with respect to these parameters. Section 2.9 deals with homogenization problem as an application of the results in the previous sections in order to find the limit of the minimal speed when the periodicity cell is very small.

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2.1 Introduction

This paper is a continuation in the study of the propagation phenomena of pulsating travelling fronts in a periodic framework corresponding to reaction-advection-diffusion equations with heterogenous KPP (Kolmogrov, Petrovsky and Piskunov) nonlinearities. We will precisely describe the *heterogenous-periodic* setting, recall the extended notion of *pulsating travelling fronts*, and then we move to announce the main results. Let us first recall some of the basic features of the *homogenous* KPP equations.

Consider the Fisher-KPP equation:

$$u_t - \Delta u = f(u) \quad \text{in } \mathbb{R}^N. \tag{2.1.1}$$

It was introduced in the celebrated papers of Fisher (1937) and in [19] originally motivated by models in biology. Here, the main assumption is that f is, say, a C^1 function satisfying

$$\begin{cases} f(0) = f(1) = 0,\ f'(1) < 0,\ f'(0) > 0, \\ f > 0 \text{ in } (0,1),\ f < 0 \text{ in } (1,+\infty), \end{cases} \tag{2.1.2}$$

$$f(s) \leq f'(0)s\,, \forall s \in [0,1]. \tag{2.1.3}$$

As examples of such nonlinearities, we have: $f(s) = s(1-s)$ and $f(s) = s(1-s^2)$.

The important feature in (2.1.1) is that this equation has a family of planar travelling fronts. These are solutions of the form

$$\begin{cases} \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ u(t,x) = \phi(x\cdot e + ct), \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{2.1.4}$$

where $e \in \mathbb{R}^N$ is a fixed vector of unit norm which is the direction of propagation, and $c > 0$ is the speed of the front. The function $\phi : \mathbb{R} \mapsto \mathbb{R}$ satisfies

$$\begin{cases} -\phi'' + c\,\phi = f(\phi), \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{2.1.5}$$

In the original paper of Kolmogorov, Petrovsky and Piskunov, it was proved that, under the above assumptions, there is a threshold value $c^* = 2\sqrt{f'(0)} > 0$ for the speed c. Namely, no fronts exist for $c < c^*$, and, for each $c \geq c^*$, there is a unique front of the type (2.1.4-2.1.5). Uniqueness is up to shift in space or time variables.

Later, the homogenous setting was extended to a general heterogenous periodic one.

The heterogenous character appeared both in the reaction-advection-diffusion equation and in the underlying domain. The general form of these equations is

$$\begin{cases} u_t = \nabla \cdot (A(z)\nabla u) \; + q(z) \cdot \nabla u + \; f(z,u), \; t \, \in \, \mathbb{R}, \; z \, \in \, \Omega, \\ \nu \cdot A \; \nabla u(t,z) = 0, \; t \, \in \, \mathbb{R}, \; z \, \in \, \partial\Omega, \end{cases} \tag{2.1.6}$$

where $\nu(z)$ is the unit outward normal on $\partial\Omega$ at the point z.

The propagation phenomena attached with equation (3.1.1) has been widely studied in many papers. Several properties of pulsating fronts in periodic media and their speed of propagation were given in several papers (Berestycki, Hamel [2], Berestycki, Hamel, Nadirashvili [3], and Berestycki, Hamel, Roques [5, 6] and Xin [36]). In section 2.2, we will recall the periodic framework and some known results which motivate our study. The main results of this paper are presented in sections 2.3 to 2.6.

2.2 The periodic framework

2.2.1 Pulsating travelling fronts in periodic domains

In this section, we introduce the general setting with the precise assumptions. Concerning the domain, let $N \geq 1$ be the space dimension, and let d be an integer so that $1 \leq d \leq N$. For an element $z = (x_1, x_2, \cdots, x_d, x_{d+1}, \cdots, x_N) \in \mathbb{R}^N$, we call $x = (x_1, x_2, \cdots, x_d)$ and $y = (x_{d+1}, \cdots, x_N)$ so that $z = (x, y)$. Let $L_1, \cdots, L_d$ be d positive real numbers, and let Ω be a C^3 nonempty connected open subset of $\mathbb{R}^N$ satisfying

$$\begin{cases} \exists\, R \geq 0\,; \forall\,(x,y) \, \in \, \Omega, \, |y| \, \leq \, R, \\ \forall\,(k_1, \cdots, k_d) \, \in \, L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}, \quad \Omega \, = \, \Omega + \displaystyle\sum_{k=1}^{d} k_i e_i, \end{cases} \tag{2.2.1}$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of $\mathbb{R}^N$. In particular, since $d \geq 1$, the set Ω is **unbounded**.

In this periodic situation, we give the following definitions:

Definition 2.2.1 (Periodicity cell) *The set* $C = \{\,(x,y) \in \Omega;\ x_1 \in (0, L_1), \cdots, x_d \in (0, L_d)\}$ *is called the periodicity cell of* Ω.

Definition 2.2.2 (L-periodic flows) *A field* $w : \Omega \to \mathbb{R}^N$ *is said to be L-periodic with respect to x if* $w(x_1 + k_1, \cdots, x_d + k_d\,, y) = w(x_1, \cdots, x_d, y)$ *almost everywhere in*

Ω, *and for all* $k = (k_1, \cdots, k_d) \in \prod_{i=1}^{d} L_i\mathbb{Z}$.

Before going further on, we point out that this framework includes several types of simpler geometrical configurations. The case of the whole space $\mathbb{R}^N$ corresponds to $d = N$, where $L_1, \ldots, L_N$ are any positive numbers. The case of the whole space $\mathbb{R}^N$ with a periodic array of holes can also be considered. The case d = 1 corresponds to domains which have only one unbounded dimension, namely infinite cylinders which may be straight or have oscillating periodic boundaries, and which may or may not have periodic holes. The case $2 \leq d \leq N-1$ corresponds to infinite slabs.

We are concerned with propagation phenomena for the reaction-advection-diffusion equation (3.1.1) set in the periodic domain Ω. Such equations arise in combustion models for flame propagation (see [27], [31] and [37]), as well as in models in biology and for population dynamics of a species (see [14], [11], [20] and [28]). These equations are used in modeling the propagation of a flame or of an epidemics in a periodic heterogenous medium. The passive quantity u typically stands for the temperature or a concentration which diffuses in a periodic excitable medium. However, in some sections we will ignore the advection and deal only with reaction-diffusion equations.

Let us now detail the assumptions concerning the coefficients in (3.1.1). First, the diffusion matrix $A(x,y) = (A_{ij}(x,y))_{1\leq i,j\leq N}$ is a *symmetric* $C^{2,\delta}(\overline{\Omega})$ (with $\delta > 0$) matrix field satisfying

$$\begin{cases} A \text{ is } L\text{-periodic with respect to } x, \\ \exists\, 0 < \alpha_1 \leq \alpha_2, \forall (x,y) \in \Omega, \forall \xi \in \mathbb{R}^N, \\ \alpha_1|\xi|^2 \leq \displaystyle\sum_{1\leq i,j\leq N} A_{ij}(x,y)\xi_i\xi_j \leq \alpha_2|\xi|^2. \end{cases} \tag{2.2.2}$$

The boundary condition $\nu{\cdot}A\nabla u(x,y) = 0$ stands for $\displaystyle\sum_{1\leq i,j\leq N} \nu_i(x,y)A_{ij}(x,y)\partial_{x_j}u(t,x,y)$, and ν stands for the unit outward normal on $\partial\Omega$. We note that when A is the identity matrix, then this boundary condition reduces to the usual Neumann condition $\partial_\nu u = 0$.

The underlying advection $q(x,y) = (q_1(x,y), \cdots, q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$)

vector field satisfying

$$\begin{cases} q \quad \text{is } L- \text{ periodic with respect to } \; x, \\ \nabla \cdot q \, = 0 \quad \text{in } \overline{\Omega}\,, \\ q \cdot \nu \, = 0 \quad \text{on } \partial\Omega\,, \\ \forall\, 1 \leq \, i \, \leq \, d, \quad \displaystyle\int_C q_i \; dx \, dy \; = \; 0. \end{cases} \tag{2.2.3}$$

Concerning the nonlinearity, let $f \; = \; f(x, y, u)$ be a nonnegative function defined in $\overline{\Omega} \, \times [0, 1]$, such that

$$\begin{cases} f \geq 0, f \text{ is } L\text{-periodic with respect to } x, \text{ and of class } C^{1,\,\delta}(\overline{\Omega} \, \times \, [0, 1]), \\ \forall\, (x, y) \, \in \, \overline{\Omega}, \quad f(x, y, 0) = \, f(x, y, 1) = \, 0, \\ \exists\, \rho \in \, (0, 1), \; \forall (x, y) \, \in \overline{\Omega}, \; \forall\, 1 - \rho \leq \, s \, \leq s' \, \leq \, 1, \; f(x, y, s) \, \geq \, f(x, y, s'), \\ \forall\, s \in \, (0, 1), \; \exists\, (x, y) \in \overline{\Omega} \; \text{ such that } f(x, y, s) \, > \, 0, \\ \forall\, (x, y) \, \in \, \overline{\Omega}, \quad f'_u(x, y, 0) \, = \, \displaystyle\lim_{u \to 0^+} \frac{f(x, y, u)}{u} \, > \, 0, \end{cases} \tag{2.2.4}$$

with the additional assumption

$$\forall\, (x, y, s) \in \overline{\Omega} \times (0, 1), \quad 0 \, < \, f(x, y, s) \, \leq \, f'_u(x, y, 0) \times \, s. \tag{2.2.5}$$

We denote by $\zeta(x, y) := \, f'_u(x, y, 0)$, for each $(x, y) \in \overline{\Omega}$.
The set of such nonlinearities contains two particular types of functions:

— The homogeneous (KPP) type: $f(x, y, u) = g(u)$, where g is a $C^{1,\delta}$ function that satisfies:
$g(0) = g(1) = \, 0, \; g > 0$ on $(0, 1), \; g'(0) > 0, \; g'(1) < 0$ and $0 < g(s) \leq g'(0)s$ in $(0, 1)$.

— Another type of such nonlinearities consists of functions $f(x, y, u) = h(x, y).\tilde{f}(u)$, such that $\tilde{f}$ is of the previous type, while h lies in $C^{1,\delta}(\overline{\Omega})$, L -periodic with respect to x, and positive in $\overline{\Omega}$.

Having this periodic framework, the notions of travelling fronts and propagation were extended, in [2], [3], [11], [26] [28], [29], and [34] as follows:

Definition 2.2.3 *Let $e \, = (e^1, \cdots, e^d)$ be an arbitrarily given vector in $\mathbb{R}^d$. A function $u = u(t, x, y)$ is called a pulsating travelling front propagating in the direction of e with* *an effective speed* $c \neq 0$*, if* u *is a classical solution of*

$$\begin{cases} u_t = \nabla\cdot(A(x,y)\nabla u) + q(x,y)\cdot\nabla u + f(x,y,u), \; t\in\mathbb{R}, \; (x,y)\in\Omega, \\ \nu\cdot A\,\nabla u(t,x,y) = 0, \; t\in\mathbb{R}, \; (x,y)\in\partial\Omega, \\ \forall\, k\in\prod_{i=1}^{d} L_i\mathbb{Z}, \; \forall\,(t,x,y)\in\mathbb{R}\times\overline{\Omega}, \quad u(t-\frac{k\cdot e}{c},x,y) = u(t,x+k,y), \\ \lim\limits_{x\cdot e\to-\infty} u(t,x,y) = 0, \text{ and } \lim\limits_{x\cdot e\to+\infty} u(t,x,y) = 1, \\ 0\le u\le 1, \end{cases} \tag{2.2.6}$$

where the above limits hold locally in t *and uniformly in* y *and in the directions of* $\mathbb{R}^d$ *which are orthogonal to* e *.*

2.2.2 Some important known results concerning the propagation phenomena in a periodic framework

Under the assumptions (3.1.2), (3.1.3), (3.1.4), (2.2.4) and (4.1.5) set in the previous subsection, Berestycki and Hamel [2] proved that: having a pre-fixed unit vector $e\in\mathbb{R}^d$, there exists $c^*(e)>0$ such that pulsating travelling fronts propagating in the direction e (i.e satisfying (4.1.8)) with a speed of propagation c exist if and only if $c\ge c^*(e)$; moreover, the pulsating fronts (within a speed $c\ge c^*(e)$) are increasing in the time t. The value $c^*(e)=c^*_{\Omega,A,q,f}(e)$ is called the *minimal speed of propagation in the direction of* e. Other nonlinearities have been considered in the cases of the whole space $\mathbb{R}^N$ or in the general periodic framework (see [2], [28], [29], [32], [33], [34], [35]).

Having the threshold value $c^*_{\Omega,A,q,f}(e)$, our paper aims to study the limits, the asymptotic behaviors, and the variations of some parametric quantities. These parametric quantities involve the parametric speeds of propagation of different reaction-advection-diffusion problems within a diffusion factor $\varepsilon>0$, a reaction factor $B>0$, or a periodicity parameter L. Thus, it is important to have a variational characterization which shows the dependance of the minimal speed of propagation on the coefficients A, q and f and on the geometry of the domain Ω. In this context, Berestycki, Hamel, and Nadirashvili [3] gave such a formulation for $c^*_{\Omega,A,q,f}(e)$ involving elliptic eigenvalue problems. We recall this variational characterization in the following theorem:

Theorem 2.2.4 (Berestycki, Hamel, and Nadirashvili [3]) *Let* e *be a fixed unit vector in* $\mathbb{R}^d$*. Let* $\tilde{e}=(e,0,\ldots,0)\in\mathbb{R}^N$*. Assume that* Ω*,* A *and* f *satisfy (3.1.2),(3.1.3), (2.2.4), and (4.1.5). The minimal speed* $c^*(e)=c^*_{\Omega,A,q,f}(e)$ *of pulsating fronts solving*

(4.1.8) and propagating in the direction of e is given by

$$c^*(e) = c^*_{\Omega,A,q,f}(e) \; = \; \min_{\lambda>0} \frac{k(\lambda)}{\lambda}, \tag{2.2.7}$$

where $k(\lambda) = k_{\Omega,e,A,q,\zeta}(\lambda)$ *is the principal eigenvalue of the operator* $L_{\Omega,e,A,q,\zeta,\lambda}$ *which is defined by*

$$\begin{aligned} L_{\Omega,e,A,q,\zeta,\lambda}\psi \; := \; & \nabla\cdot(A\nabla\psi) - 2\lambda\tilde{e}\cdot A\nabla\psi + q\cdot\nabla\psi \\ & +[\lambda^2\tilde{e}A\tilde{e} - \lambda\nabla\cdot(A\tilde{e}) - \lambda q\cdot\tilde{e} + \zeta]\psi \end{aligned} \tag{2.2.8}$$

acting on the set

$$E = \; \{\, \psi\in C^2(\overline{\Omega}), \psi \text{ is } L\text{-periodic with respect to } x \text{ and } \nu\cdot A\nabla\psi = \lambda(\nu A\tilde{e}\psi) \text{ on } \partial\Omega \,\}\,.$$

The proof of formula (3.1.18) is based on methods developed in [2], [7] and [9]. These are techniques of sub and super-solutions, regularizing and approximations in bounded domains.

We note that in formula (3.1.18), the value of the minimal speed $c^*(e)$ is given in terms of the direction e, the domain Ω, and the coefficients A, q and $f^{'}_u(.,.,0)$. Moreover, it is important to notice that the dependence of $c^*(e)$ on the nonlinearity f is only through the derivative of f with respect to u at $u = 0$.

Before going further on, let us mention that formula (3.1.18) extends some earlier results about front propagation. When $\Omega = \mathbb{R}^N$, $A = Id$ and $f = f(u)$ (with $f(u) \leq f^{'}(0)u$ in $[0,1]$), formula (3.1.18) then reduces to the well-known KPP formula $c^*(e) = 2\sqrt{f^{'}(0)}$. That is the value of the minimal speed of propagation of planar fronts for the *homogenous* reaction-diffusion equation: $u_t - \Delta u = f(u)$ in $\mathbb{R}^N$. [2]

The above variational characterization of the minimal speed of propagation of pulsating fronts in general periodic excitable media will play the main role in studying the dependence of the minimal speed $c^*(e) = c^*_{\Omega,A,q,f}(e)$ on the coefficients of reaction, diffusion, advection and on the geometry of the domain. In this context, we have:

Theorem 2.2.5 (Berestycki, Hamel, Nadirashvili [3]) *Under the assumptions (3.1.2), (3.1.3), and (3.1.4) on* Ω*,* A*, and* q*, let* $f = f(x,y,u)$ *[respectively* $g = g(x,y,u)$*] be*

2. In fact, the uniqueness, up to multiplication by a non-zero real number, of the first eigenvalue function of $L_{\mathbb{R}^N,e,Id,f^{'}(0),\lambda}\psi = k(\lambda)\psi$ together with this particular situation, yield that the principal eigenfunction ψ is constant and $k(\lambda) = \lambda^2 + f^{'}(0)$ for all $\lambda > 0$. Therefore by (3.1.18), we have $c^*(e) = \min_{\lambda>0}\left(\lambda + \dfrac{f^{'}(0)}{\lambda}\right) = 2\sqrt{f^{'}(0)}$.

a nonnegative nonlinearity satisfying (2.2.4) and (4.1.5). Let e be a fixed unit vector in $\mathbb{R}^d$, where $1 \leq d \leq N$,

a) If $f_u'(x,y,0) \leq g_u'(x,y,0)$ for all $(x,y) \in \overline{\Omega}$, then

$$c^*_{\Omega,A,q,f}(e) \leq c^*_{\Omega,A,q,g}(e).$$

Moreover if $f_u'(x,y,0) \leq, \not\equiv g_u'(x,y,0)$ in $\overline{\Omega}$, then $c^*_{\Omega,A,q,f}(e) < c^*_{\Omega,A,q,g}(e)$.

b) The map $B \mapsto c^*_{\Omega,A,q,Bf}(e)$ is increasing in $B > 0$ and

$$\limsup_{B \to +\infty} \frac{c^*_{\Omega,A,q,Bf}(e)}{\sqrt{B}} < +\infty.$$

Furthermore, if $\Omega = \mathbb{R}^N$ or if $\nu A\tilde{e} \equiv 0$ on $\partial\Omega$, then $\liminf\limits_{B \to +\infty} \dfrac{c^*_{\Omega,A,q,Bf}(e)}{\sqrt{B}} > 0$.

c)

$$c^*_{\Omega,A,q,f}(e) \leq ||(q.\tilde{e})^-||_\infty + 2\sqrt{\max_{(x,y)\in\overline{\Omega}} \zeta(x,y)}\sqrt{\max_{(x,y)\in\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}}, \tag{2.2.9}$$

where $||(q.\tilde{e})^-||_\infty = \max\limits_{(x,y)\in\overline{\Omega}} (q(x,y).\tilde{e})^-$ and $s^- = \max(-s,0)$ for each $s \in \mathbb{R}$. Furthermore, the equality holds in (2.2.9) if and only if $\tilde{e}A\tilde{e}$ and ζ are constant, $q.\tilde{e} \equiv \nabla.(A\tilde{e}) \equiv 0$ in $\overline{\Omega}$ and $\nu.A\tilde{e} = 0$ on $\partial\Omega$ (in the case when $\partial\Omega \neq \emptyset$).

d) Assume furthermore that $f = f(u)$ and $q \equiv 0$ in $\overline{\Omega}$, then the map $\beta \mapsto c^*_{\Omega,\beta A,0,f}(e)$ is increasing in $\beta > 0$.

As a corollary of (2.2.9), we see that $\limsup\limits_{M\to+\infty} \dfrac{c^*_{\Omega,MA,q,f}(e)}{\sqrt{M}} \leq C$ where C is a positive constant. Furthermore, part d) implies that a larger diffusion speeds up the propagation in the absence of the advection field.

We mention that the existence of pulsating travelling fronts in space-time periodic media was proved in Nolen, Xin [23, 24], Nolen, Rudd, Xin [25] and recently in Nadin [21, 22]. In [22], Nadin characterized the minimal speed of propagation and he studied the influence of the diffusion, the amplitude of the reaction term and the drift on the characterized speed.

After reviewing some results in the study of the KPP propagation phenomena in a periodic framework, we pass now to announce new results concerning the limiting behavior of the minimal speed of propagation within a small (resp. large) diffusion and reaction coefficients (in some particular situations of the general periodic framework) and we will study the minimal speed as a function of the period of the coefficients in the KPP reaction-diffusion-advection (or reaction-diffusion) equation in the case where $\Omega = \mathbb{R}^N$. The proofs will be shown in details in section 2.8. The announced results will be applied to find the homogenization limit of the minimal speeds of propagation. We believe that this limit might help to find the homogenized equation in the "KPP" periodic framework (see section 2.9 for more details).

2.3 The minimal speed within small diffusion factors or within large period coefficients

In this section, our problem is a reaction-diffusion equation with absence of advection terms:

$$\begin{cases} u_t = \beta\,\nabla\cdot(A(x,y)\nabla u) \,+\, f(x,y,u),\ t\in\mathbb{R},\ (x,y)\in\Omega, \\ \nu\cdot A\,\nabla u(t,x,y) = 0,\ t\in\mathbb{R},\ (x,y)\in\partial\Omega, \end{cases} \tag{2.3.1}$$

where $\beta > 0$.

We mention that (2.3.1) is a reaction-diffusion problem within a diffusion matrix βA. Let e be a unit direction in $\mathbb{R}^d$. Under the assumptions (3.1.2), (3.1.3), (2.2.4) and (4.1.5), for each $\beta > 0$, there corresponds a minimal speed of propagation $c^*_{\Omega,\beta A,0,f}(e)$ so that a pulsating front with a speed c and satisfying (2.3.1) exists if and only if $c \geq c^*_{\Omega,\beta A,0,f}(e)$.

Referring to part c) of Theorem 2.2.5, one gets $0 < c^*_{\Omega,\beta A,0,f}(e) \leq 2\sqrt{\beta}\sqrt{M_0 M}$, for any $\beta > 0$, where $M_0 = \max\limits_{(x,y)\,\in\,\overline{\Omega}} \zeta(x,y)$ and $M = \max\limits_{(x,y)\,\in\,\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}\,.$
Consequently, there exists $C > 0$ and independent of β such that

$$\forall\beta > 0,\ 0 < \frac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}} \leq C. \tag{2.3.2}$$

The inequality (2.3.2) leads us to investigate the limits of $\dfrac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}}$ as $\beta \to 0$ and as $\beta \to +\infty$. The following theorem gives the precise limit when the diffusion factor tends to zero. However, it will not be announced in the most general periodic setting. We will describe the situation before the statement of the theorem:

The domain will be in the form $\Omega = \mathbb{R}\times\omega \subseteq \mathbb{R}^N$, where $\omega \subseteq \mathbb{R}^d\times\mathbb{R}^{N-d-1}$ $(d \geq 0)$. If $d = 0$, then ω is a C^3 connected, open bounded subset of $\mathbb{R}^{N-1}$. While, in the case where $1 \leq d \leq N-1$, ω is a $(L_1,\ldots,L_d)$-periodic open domain of $\mathbb{R}^{N-1}$ which satisfies (3.1.2); and hence, Ω is a $(l, L_1,\ldots,L_d)-$periodic subset of $\mathbb{R}^N$ that satisfies (3.1.2) with $l > 0$ and arbitrary. An element of $\Omega = \mathbb{R} \times \omega$ will be represented as $z = (x, y)$ where $x \in \mathbb{R}$ and $y \in \omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-1-d}$.

The nonlinearity $f = f(x, y, u)$, in this section, is a KPP nonlinearity defined on $\overline{\Omega} \times [0, 1]$ that satisfies

$$\begin{cases} f \geq 0, \text{ and of class } C^{1,\delta}(\mathbb{R} \times \overline{\omega} \times [0,1]), \\ f \text{ is } (l, L_1, \ldots, L_d)\text{-periodic with respect to } (x, y_1, \ldots, y_d), \text{ when } d \geq 1, \\ f \text{ is } l\text{-periodic in } x, \text{ when } d = 0, \\ \forall\, (x, y) \in \overline{\Omega} = \mathbb{R} \times \overline{\omega},\ f(x, y, 0) = f(x, y, 1) = 0, \\ \exists\, \rho \in (0, 1),\ \forall (x, y) \in \overline{\Omega},\ \forall\, 1 - \rho \leq s \leq s' \leq 1,\ f(x, y, s) \geq f(x, y, s'), \\ \forall\, s \in (0, 1), \quad \exists\, (x, y) \in \overline{\Omega} \text{ such that } f(x, y, s) > 0, \end{cases} \tag{2.3.3}$$

together with the assumptions

$$\begin{cases} f'_u(x, y, 0) \text{ depends only on } y;\ \text{we denote by } \zeta(y) = f'_u(x, y, 0),\ \forall (x, y) \in \overline{\Omega}. \\ \forall\, (x, y) \in \overline{\Omega} = \mathbb{R} \times \overline{\omega}, \quad f'_u(x, y, 0) = \zeta(y) > 0, \\ \forall\, (x, y, s) \in \overline{\Omega} \times (0, 1), \quad 0 < f(x, y, s) \leq \zeta(y)\, s. \end{cases} \tag{2.3.4}$$

Notice that $f'_u(x, y, u)$ is assumed to depend only on y, but $f(x, y, u)$ may depend on x.

Lastly, concerning the diffusion matrix, $A(x, y) = A(y) = (A_{ij}(y))_{1 \leq i,j \leq N}$ is a $C^{2,\delta}(\overline{\Omega})$ (with $\delta > 0$) symmetric matrix field whose entries are depending only on y, and satisfying

$$\begin{cases} A \text{ is } (L_1, \ldots, L_d)\text{-periodic with respect to } (y_1, \ldots, y_d), \\ \exists\, 0 < \alpha_1 \leq \alpha_2,\ \forall\, y \in \omega, \forall \xi \in \mathbb{R}^N, \\ \alpha_1 |\xi|^2 \leq \sum A_{ij}(y) \xi_i \xi_j \leq \alpha_2 |\xi|^2. \end{cases} \tag{2.3.5}$$

Theorem 2.3.1 *Let $e = (1, 0, \ldots, 0) \in \mathbb{R}^N$ and $\varepsilon > 0$. Let $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$ satisfy the form described in the previous page. Under the assumptions (2.3.3), (2.3.4), and (2.3.5), consider the reaction-diffusion equation*

$$\begin{cases} u_t(t, x, y) = \varepsilon\, \nabla \cdot (A(y) \nabla u)(t, x, y) + f(x, y, u), \ \textit{for}\ (t, x, y) \in \mathbb{R} \times \Omega, \\ \nu \cdot A \nabla u = 0 \quad \textit{on} \quad \mathbb{R} \times \mathbb{R} \times \partial\omega. \end{cases} \tag{2.3.6}$$

Assume, furthermore, that A and f satisfy one of the following two alternatives:

$$\begin{cases} \exists\,\alpha > 0,\ \forall y \in \omega,\ A(y)e = \alpha e, \\ f_u'(x,y,0) = \zeta(y),\ \textit{for all}\ (x,y) \in \overline{\Omega}, \end{cases} \tag{2.3.7}$$

or

$$\begin{cases} f_u'(x,y,0) = \zeta\ \ \textit{is constant}, \\ \forall y \in \omega,\ A(y)e = \alpha(y)e,\ \ \textit{where} \\ \ y \mapsto \alpha(y)\ \textit{is a positive},\ (L_1,\dots,L_d)-\textit{periodic function over}\ \overline{\omega}. \end{cases} \tag{2.3.8}$$

Then,

$$\lim_{\varepsilon\to0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \;=\; 2\sqrt{\max_{\overline{\omega}}\,\zeta}\sqrt{\max_{\overline{\omega}}\ eAe}. \tag{2.3.9}$$

Before going further on, we mention that the family of domains for which Theorem 2.3.1 holds is wide. An infinite cylinder $\mathbb{R}\times B_{\mathbb{R}^{N-1}}(y_0,R)$ (where $R>0$, and $B_{\mathbb{R}^{N-1}}(y_0,R)$ is the Euclidian ball of center y_0 and radius R) is an archetype of such domains. In these cylinders, $\omega = B_{\mathbb{R}^{N-1}}(y_0,R)$, l is any positive real number, and $d=0$. The whole space $\mathbb{R}^N$ is another archetype of the domain Ω where $d=N-1$, $\omega=\mathbb{R}^{N-1}$, and $\{l,\,L_1,\dots,L_d\}$ is any family of positive real numbers.

Remark 2.3.2 In Theorem 2.3.1, the domain $\Omega=\mathbb{R}\times\omega$ is invariant in the direction of $e=(1,0\dots,0)$ which is parallel to Ae (in both cases (2.3.7) and (2.3.8)). Also, the assumption that the entries of A do not depend on x, yields that $\nabla.(Ae)\equiv 0$ over Ω. On the other hand, it is easy to find a diffusion matrix A and a nonlinearity f which satisfy, together, the assumptions of Theorem 2.3.1 while one of $eAe(y)$ and $\zeta(y)$ is **not constant**. Referring to part c) of Theorem 2.2.5, one obtains:

$$\forall\varepsilon>0,\quad 0<\frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \lneqq 2\sqrt{\max_{y\in\overline{\omega}}\,\zeta(y)}\sqrt{\max_{y\in\overline{\omega}}\,eAe(y)}.$$

However, Theorem 2.3.1 implies that

$$\lim_{\varepsilon\to0^+}\frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \;=\; 2\sqrt{\max_{y\in\overline{\omega}}\,\zeta(y)}\sqrt{\max_{y\in\overline{\omega}}\,eAe(y)}.$$

On the other hand, if $\Omega=\mathbb{R}\times\omega$ as in Theorem 2.3.1, $A=Id$ and $f=f(u)$, Theorem 2.2.5 yields that $c^*_{\Omega,\varepsilon Id,0,f}(e) = 2\sqrt{\varepsilon}\sqrt{f'(0)}$, for all $\varepsilon>0$. $\square$

In the same context, one can also find the limit when the diffusion factor goes to zero, but in the presence of an advection field in the form of shear flows:

Theorem 2.3.3 *Assume that* $e = (1, 0, \cdots, 0) \in \mathbb{R}^N$, *the domain* $\Omega = \mathbb{R} \times \omega$ *has the same form as in Theorem 2.3.1, and the coefficients* f *and* A *satisfy (2.3.3-2.3.4) and (2.3.5) respectively. Assume, furthermore, that for all* $y \in \overline{\omega}$, *there exists* $\alpha(y)$ *positive so that* $A(y)e = \alpha(y)e$ *in* $\overline{\omega}$. *Consider, in addition, an advective shear flow* $q = (q_1(y), 0, \ldots, 0)$ $(y \in \overline{\omega})$ *which is* $(L_1, \cdots, L_d)-$*periodic with respect to* y. *Assume that* ε *is a positive parameter and consider the parametric reaction-advection-diffusion problem*

$$\begin{cases} u_t = \varepsilon \nabla \cdot (A(y)\nabla u) + q_1(y)\, \partial_x u(t,x,y) + f(x,y,u), & t \in \mathbb{R},\ (x,y) \in \Omega, \\ \nu \cdot A\, \nabla u(t,x,y) = 0, \quad t \in \mathbb{R}, & (x,y) \in \partial\Omega, \end{cases} \tag{2.3.10}$$

where $q \not\equiv 0$ *over* $\mathbb{R} \times \overline{\omega}$ *and* q *has a zero average. Then,*

$$\lim_{\varepsilon \to 0^+} c^*_{\Omega, \varepsilon A, q, f}(e) = \max_{y \in \overline{\omega}} \,(-\, q_1(y)) = \max_{\overline{\omega}}(-\, q.e). \tag{2.3.11}$$

Moreover, in the same setting ***together with the additional assumptions:*** <u>eAe ***and*** ζ ***are both co***</u>
one has

$$\lim_{\varepsilon \to 0^+} \frac{c^*_{\Omega, \varepsilon A, \sqrt{\varepsilon}\, q, f}(e)}{\sqrt{\varepsilon}} = \max_{\overline{\omega}}(-q(y).e) + 2\sqrt{\alpha}\sqrt{\zeta_0}, \tag{2.3.12}$$

where $\zeta(y) = \zeta_0$ *and* $eA(y)e = \alpha$ *for all* $y \in \overline{\omega}$.

The situation in this result is more general than that considered in part b) of Corollary 4.5 in [4]. In details, the coefficients A and f can be both non-constant. Meanwhile, in the result of [4], the coefficients considered were assumed to satisfy the alternative (2.3.7).

After having the exact value of $\lim\limits_{\varepsilon \to 0^+} \dfrac{c^*_{\Omega, \varepsilon A, 0, f}(e)}{\sqrt{\varepsilon}}$, we move now to investigate the limit of the minimal speed of propagation, considered as a function of the period of the coefficients of the reaction-diffusion equation set in the whole space $\mathbb{R}^N$, when the periodicity parameter tends to $+\infty$. By making some change in variables, we will find a link between this problem and Theorem 2.3.1:

Theorem 2.3.4 *Let* $e = (1, 0, \ldots, 0) \in \mathbb{R}^N$. *An element* $z \in \mathbb{R}^N$ *is represented as* $z = (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$. *Assume that* $f = f(x, y, u)$ *and* $A = A(y)$ *satisfy (2.3.3), (2.3.4) and (2.3.5) with* $\omega = \mathbb{R}^{N-1}$, $d = N - 1$, *and* $l = L_1 = \ldots = L_{N-1} = 1$. *(That is, the domain and the coefficients of the equation are* $(1, 1, \ldots, 1)$ *periodic with respect to* y*). Assume furthermore, that* A *and* f *satisfy either (2.3.7) or (2.3.8). For each* $L > 0$, *and* $(x, y) \in \mathbb{R}^N$, *let* $A_L(y) = A(\dfrac{y}{L})$ *and* $f_L(x, y, u) = f(\dfrac{x}{L}, \dfrac{y}{L}, u)$. *Consider the* *reaction-diffusion problem*

$$\begin{aligned} u_t(t,x,y) &= \nabla\cdot(A_{_L}\nabla u)(t,x,y) \,+\, f_{_L}(x,y,u),\ (t,x,y)\in\mathbb{R}\times\mathbb{R}^N \\ &= \nabla\cdot(A(\frac{y}{L})\nabla u)(t,x,y) \,+\, f(\frac{x}{L},\frac{y}{L},u),\ (t,x,y)\in\mathbb{R}\times\mathbb{R}^N, \end{aligned} \tag{2.3.13}$$

whose coefficients are $(L,\ldots,L)$ *periodic with respect to* $(x,y)\in\mathbb{R}^N$*. Then,*

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N,\,A_{_L},\,0,\,f_{_L}}(e) = 2\sqrt{\max_{y\,\in\mathbb{R}^{N-1}}\zeta(y)}\sqrt{\max_{y\,\in\mathbb{R}^{N-1}} e.Ae(y)}. \tag{2.3.14}$$

The above theorem gives the limit of the minimal speed of propagation in the direction of $e=(1,0,\cdots,0)$ as the periodicity parameter $L\to+\infty$. The domain is the whole space $\mathbb{R}^N$ which is $(L,\cdots,L)-$periodic whatever the positive number L. However, one can find

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N,\,A_{_L},\,Lq_{_L},\,f_{_L}}(e) \text{ and } \lim_{L\to+\infty} c^*_{\mathbb{R}^N,\,A_{_L},\,q_{_L},\,f_{_L}}(e)$$

whenever q is a shear flow advection. Namely, in the same manner that Theorem 2.3.1 implies Theorem 2.3.4, one can prove that Theorem 2.3.3 implies

Theorem 2.3.5 *Let* $e=(1,0,\ldots,0)\in\mathbb{R}^N$*. Assume that* $f=f(x,y,u)$ *and* $A=A(y)$ *satisfy (2.3.3), (2.3.4) and (2.3.5) with* $\omega=\mathbb{R}^{N-1}$*,* $d=N-1$*, and* $l=L_1=\ldots=L_{N-1}=1$*. (That is, the domain and the coefficients of the equation are* $(1,1,\ldots,1)$ *periodic with respect to* y *in* $\mathbb{R}^{N-1}$*). Assume, furthermore, that for all* $y\in\mathbb{R}^{N-1}$*, there exists* $\alpha(y)$ *positive so that* $A(y)e=\alpha(y)e$ *in* $\mathbb{R}^{N-1}$*. Let* $q=(q_{_1}(y),0,\ldots,0)$ *for all* $y\in\mathbb{R}^{N-1}$ *such that* $q_1\not\equiv 0$ *over* $\mathbb{R}^{N-1}$*,* q *is* $(1,\cdots,1)-$*periodic with respect to* y *and* q_1 *has a zero average. Then,*

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N,\,A_{_L},\,Lq_{_L},\,f_{_L}}(e) = \max_{y\in\mathbb{R}^{N-1}}(-\,q_{_1}(y)) = \max_{y\in\mathbb{R}^{N-1}}(-\,q(y).e). \tag{2.3.15}$$

Moreover, if eAe and ζ *are both constant over* $\mathbb{R}^{N-1}$*, then*

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N,\,A_{_L},\,q_{_L},\,f_{_L}}(e) = \max_{y\in\mathbb{R}^{N-1}}(-\,q(y).e) + 2\sqrt{\alpha}\sqrt{\zeta_0}, \tag{2.3.16}$$

where $\zeta_0=\zeta(y)$ *and* $\alpha=eA(y)e$ *for all* $y\in\mathbb{R}^{N-1}$*.*

In the proof of Theorem 2.3.3 (which implies Theorem 2.3.5), the assumption that the advection q is in the form of shear flows plays an important role in reducing the elliptic equation involved by the variational formula (2.8.13) below. Namely, since $q=$

$(q_1(y), 0, \cdots, 0)$ and since $e = (1, 0, \cdots, 0)$, then the terms $q(x, y) \cdot \nabla_{x,y}\psi$ and $q(x, y) \cdot e$ (in the general elliptic equation) become equal to $q_1(y)\partial_x\psi$ and $q_1(y)$ respectively. As a consequence, and due the uniqueness of the principal eigenfunction ψ up to multiplication by a constant, we are able to choose ψ independent of x, and hence, obtain a symmetric elliptic operator (without drift) whose principal eigenvalue was given by the variational formula (2.8.15) below (see section 2.8 for more details).

Remark 2.3.6 *After the above explanations, we find that the techniques used to prove Theorem 2.3.3 which implies 2.3.5, will no longer work in the presence a general periodic advection field satisfying (3.1.4).*

Concerning the influence of advection, we mention that the limit of $\dfrac{c^*_{\Omega,A,\,Bq,f}(e)}{B}$ as $B \to +\infty$ (in the general periodic setting) is not yet given explicitly as a function of the direction e and the coefficients A, q and f. For more details one can see Theorem 4.1 in [4]. However, the problem of front propagation in an infinite cylinder with an underlying shear flow was widely studied in Berestycki [1], Berestycki and Nirenberg [8]. In the case of strong advection, assume that $\Omega = \mathbb{R} \times \omega$, where ω is a bounded smooth subset of $\mathbb{R}^{N-1}$, $q = (q_{_1}(y), 0, \cdots, 0)$, $y \in \omega$, and $f = f(u)$ is a (KPP) nonlinearity. It was proved, in Heinze [16], that

$$\lim_{B\to+\infty} \frac{c^*_{\Omega,A,\,Bq,f}(e)}{B} = \gamma, \tag{2.3.17}$$

where

$$\gamma = \sup_{\psi \in D} \int_\omega q_{_1}(y)\,\psi^2\,dy,$$

$$D = \left\{ \psi \in H^1(w), \quad \int_\omega |\nabla\psi|^2\,dy \,\le\, f^{'}(0), \text{ and } \int_\omega \psi^2\,dy = 1 \right\}.$$

2.4 The minimal speed within large diffusion factors or within small period coefficients

After having the limit of $c^*_{\Omega,\varepsilon A,0,f}(e)/\sqrt{\varepsilon}$ as $\varepsilon \to 0^+$, and after knowing that this limit depends on $\max\limits_{y\in\overline{w}} \zeta(y)$ and $\max\limits_{y\in\overline{w}} eAe(y)$, we investigate now the limit of $c^*_{\Omega,MA,\,q,f}(e)/\sqrt{M}$ as the diffusion factor M tends to $+\infty$, and we try to answer this question in a situation which is more general than that we considered in the previous section (in the case where the diffusion factor was going to 0^+). That is in the presence of an advection field and in a domain Ω which satisfies (3.1.2) and which may take more forms other than those of section 2.3. We will find that in the case of large diffusion, the limit will depend on $\displaystyle -\!\!\!\!\!\!\int_C \zeta(x,y)dx\,dy := \frac{1}{|C|}\int_C \zeta(x,y)dx\,dy$ and $\displaystyle -\!\!\!\!\!\!\int_C \tilde{e}A\tilde{e}(x,y)dx\,dy := \frac{1}{|C|}\int_C \tilde{e}A\tilde{e}(x,y)dx\,dy,$ where C denotes the periodicity cell of the domain Ω.

Theorem 2.4.1 *Under the assumptions (3.1.2) for Ω, (3.1.4) for the advection q, (2.2.4) and (4.1.5) for the nonlinearity $f = f(x,y,u)$, let e be any unit direction of $\mathbb{R}^d$. Assume that the diffusion matrix $A = A(x,y)$ satisfies (3.1.3) together with $\nabla \cdot A\tilde{e} \equiv 0$ over Ω, and $\nu \cdot A\tilde{e} = 0$ over $\partial\Omega$. For each $M > 0$ and $0 \leq \gamma \leq 1/2$, consider the following reaction-advection-diffusion equation*

$$\begin{cases} u_t = M\,\nabla\cdot(A(x,y)\nabla u) \;+\; M^{\gamma}\,q(x,y)\cdot\nabla u \;+\; f(x,y,u),\; t\,\in\,\mathbb{R},\;(x,y)\,\in\,\Omega, \\ \nu\cdot A\,\nabla u(t,x,y) = 0,\; t\,\in\,\mathbb{R},\;(x,y)\,\in\,\partial\Omega. \end{cases}$$

Then

$$\lim_{M\to+\infty}\frac{c^*_{\Omega,MA,\,M^{\gamma}q,\,f}(e)}{\sqrt{M}} = 2\sqrt{-\!\!\!\!\!\!\int_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\,\sqrt{-\!\!\!\!\!\!\int_C \zeta(x,y)dx\,dy},$$

where C is the periodicity cell of Ω.

Remarks 2.4.2

— The setting in Theorem 2.4.1 is more general than that in Theorem 2.3.1, where: $\Omega = \mathbb{R}\times\omega$, $\tilde{e} = (1,0,\ldots,0)$, and $A\tilde{e} = \alpha(y)\tilde{e}$. Under the assumptions of Theorem 2.3.1, the domain Ω is invariant in the direction of $A\tilde{e}$, which is that of $\tilde{e}$. Consequently, if ν denotes the outward normal on $\partial\Omega = \mathbb{R}\times\partial\omega$, one gets $\nu\cdot A\tilde{e} = \alpha(y)\,\nu\cdot\tilde{e} = 0$ over $\partial\Omega$, while $\nabla\cdot(A\tilde{e}) = \dfrac{\partial}{\partial x}\alpha(y) = 0$ over Ω. Moreover, in Theorem 2.3.1, we have only reaction and diffusion terms. That is $q \equiv 0$. Therefore, considering the setting of Theorem 2.3.1, and taking βA as a parametric diffusion matrix, one consequently knows the limits of $\dfrac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}}$ as $\beta\to 0^+$ (Theorem 2.3.1) and as $\beta\to+\infty$ (Theorem 2.4.1).

— The other observation in Theorem 2.4.1 is that the limit does not depend on the advection field q. This may play an important role in drawing counterexamples to answer many different questions. For example, the variation of the minimal speed of propagation with respect to the diffusion factor and with respect to diffusion matrices which are symmetric positive definite.

— Another important feature, in Theorem 2.4.1, is that the order of M in the denominator of the ratio $\dfrac{c^*_{\Omega,MA,\,M^{\gamma}q,\,f}(e)}{\sqrt{M}}$ is equal to $1/2$. It is independent of γ. Consequently, **the case where the advection is null** and there is only a reaction-diffusion equation follows, in particular, from the previous theorem. That is

$$\lim_{M\to+\infty}\frac{c^*_{\Omega,MA,0,f}(e)}{\sqrt{M}}=2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\,\sqrt{\fint_C \zeta(x,y)dx\,dy}.$$

— The previous point leads us to conclude that the presence of an advection with a factor M^γ, where $0\le\gamma\le 1/2$, will have no more effect on the ratio $\dfrac{c^*_{\Omega,MA,\,M^\gamma q,\,f}(e)}{\sqrt{M}}$ as soon as the diffusion factor M gets very large.

As far as the limit of the minimal speed of propagation within small periodic coefficients in the reaction-diffusion equation is concerned, the following theorem, which mainly depends on Theorem 2.4.1, treats this problem:

Theorem 2.4.3 *Let $\Omega=\mathbb{R}^N$. Assume that $A=A(x,y)$, $q=q(x,y)$ and $f=f(x,y,u)$ are $(1,\ldots,1)-$periodic with respect to $(x,y)\in\mathbb{R}^N$, and that they satisfy (3.1.3), (3.1.4), (2.2.4),and (4.1.5) with $L_1=\ldots=L_N=1$. Let e be any unit direction of $\mathbb{R}^N$, such that $\nabla\cdot A\tilde{e}\equiv 0$ over $\mathbb{R}^N$. For each $L>0$, let $A_L(x,y)=A(\frac{x}{L},\frac{y}{L})$, $q_L(x,y)=q(\frac{x}{L},\frac{y}{L})$, and $f_L(x,y,u)=f(\frac{x}{L},\frac{y}{L},u)$, where $(x,y)\in\mathbb{R}^N$. Consider the problem*

$$\begin{aligned}u_t(t,x,y)&=\nabla\cdot(A_L\nabla u)(t,x,y)\,+\,q_L\cdot\nabla u(t,x,y)+f_L(x,y,u),\ (t,x,y)\in\mathbb{R}\times\mathbb{R}^N,\\&=\nabla\cdot(A(\frac{x}{L},\frac{y}{L})\nabla u)(t,x,y)\,+\,q(\frac{x}{L},\frac{y}{L})\cdot\nabla u(t,x,y)+f(\frac{x}{L},\frac{y}{L},u),\end{aligned}\tag{2.4.1}$$

whose coefficients are $(L,\ldots,L)$ periodic with respect to $(x,y)\in\mathbb{R}^N$. Then,

$$\lim_{L\to 0^+}c^*_{\mathbb{R}^N,\,A_L,\,q_L,\,f_L}(e)=2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\,\sqrt{\fint_C \zeta(x,y)dx\,dy},$$

where, in this setting, $C=[0,1]\times\cdots\times[0,1]\subset\mathbb{R}^N$.

The above result gives the limit in any space dimension. It depends on the assumption $\nabla\cdot(A\tilde{e})\equiv 0$ in $\mathbb{R}^N$. However, if one takes $N=1$, and denotes the diffusion coefficient by $a=a(x)$, $x\in\mathbb{R}$, then the previous result holds under the assumptions that a satisfies (3.1.3) and $da/dx\equiv 0$ in $\mathbb{R}$. In other words, it holds when a is a positive constant. Thus, it is be interesting to mention that, in the one-dimensional case, the above limit was given in [13] and [17] within a general diffusion coefficient (which may be not constant over $\mathbb{R}$). In details, assume that $f=f(x,u)=(\zeta(x)-u)u$ is a 1-periodic (KPP) nonlinearity satisfying (2.2.4) with (4.1.5), and $\mathbb{R}\ni x\mapsto a(x)$ is a

$1-$periodic function which satisfies $0 < \alpha_1 \leq a(x) \leq \alpha_2$, for all $x \in \mathbb{R}$, where α_1 and α_2 are two positive constants. For each $L > 0$, consider the reaction-diffusion equation

$$\partial_t u(t,x) = \frac{\partial}{\partial x}\left(a(\frac{x}{L})\frac{\partial u}{\partial x}\right)(t,x) \; + \; \left[\zeta(\frac{x}{L}) - u(t,x)\right]u(t,x) \quad \text{for} \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.4.2}$$

It was derived in [13] and, formally, in [17] that

$$\lim_{L\to 0^+} c^*_{\mathbb{R},\,a_L,\,0,\,f_L}(e) = 2\sqrt{<a>_H \cdot \int_0^1 \zeta(x)}, \tag{2.4.3}$$

where $<a>_H$ denotes the harmonic mean of the map $x \mapsto a(x)$ over $[0,1]$.

2.5 The minimal speed within small or large reaction coefficients

In this section, the parameter of the reaction-advection-diffusion problem is the coefficient B multiplied by the nonlinearity f. In fact, it follows from Theorem 1.6 in Berestycki, Hamel and Nadirashvili [3] (recalled via Theorem 2.2.5 in the present paper) that the map $B \mapsto c^*_{\Omega,A,q,Bf}(e)/\sqrt{B}$ remains, with the assumption $\nu.A\tilde{e} = 0$ on $\partial\Omega$, bounded by two positive constants as B gets very large. Therefore, it is interesting to find the limit of $c^*_{\Omega,A,q,Bf}(e)/\sqrt{B}$ as $B \to +\infty$ even in some particular situations. Moreover, it is important to find the limit of the same quantity as $B \to 0^+$. We start with the case where $B \to +\infty$ and then we move to that where $B \to 0^+$.

Theorem 2.5.1 *Let $e = (1,0,\dots,0) \in \mathbb{R}^N$ and $B > 0$. Assume that $\Omega = \mathbb{R}\times\omega \subseteq \mathbb{R}^N$, A, and f satisfy the same assumptions of Theorem 2.3.1. That is, f and A satisfy (2.3.3), (2.3.4), and (2.3.5), and one of the two alternatives (2.3.7)-(2.3.8). Consider the reaction-diffusion equation*

$$\begin{cases} u_t(t,x,y) = & \nabla\cdot(A(y)\nabla u)(t,x,y) + B\,f(x,y,u), \; \textit{for} \; (t,x,y) \in \mathbb{R}\times\Omega, \\ \nu\cdot A\nabla u = & 0 \quad \textit{on} \quad \mathbb{R}\times\mathbb{R}\times\partial\omega. \end{cases} \tag{2.5.1}$$

Then,

$$\lim_{B\to+\infty} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} = 2\sqrt{\max_{y\in\overline{\omega}} \zeta(y)}\sqrt{\max_{y\in\overline{\omega}} eAe(y)}. \tag{2.5.2}$$

We mention that one can find the coefficients A, and f and the domain Ω of the problem (2.5.1) satisfying all the assumptions of Theorem 2.5.1, which are the same of Theorem 2.3.1, including one of the alternatives (2.3.7)-(2.3.8) while one of ζ and eAe is **not constant**. Owing to Theorem 1.10 in [3], it follows that

$$\forall B>0,\quad c^*_{\Omega,A,0,Bf}(e)\lneqq 2\sqrt{B}\sqrt{\max_{y\in\overline{\omega}}\zeta(y)}\,\sqrt{\max_{y\in\overline{\omega}}eAe(y)},$$

which is equivalent to saying that

$$\frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}\lneqq 2\sqrt{\max_{y\in\overline{\omega}}\zeta(y)}\sqrt{\max_{y\in\overline{\omega}}eAe(y)}.$$

Therefore, there are *heterogeneous* settings in which the result found in Theorem 2.5.1 does not follow trivially.

We move now to study the limit when the reaction factor B tends to 0^+. However, the situation will be more general than that in Theorem 2.5.1 because it will consider reaction-advection-diffusion equations rather than considering reaction-diffusion equations only:

Theorem 2.5.2 *Under the assumptions (3.1.2) for* Ω, *(3.1.4) for the advection* q, *(2.2.4) and (4.1.5) for the nonlinearity* $f=f(x,y,u)$, *let* e *be any unit direction of* $\mathbb{R}^d$. *Assume that the diffusion matrix* $A=A(x,y)$ *satisfies (3.1.3) together with* $\nabla\cdot A\tilde{e}\equiv 0$ *over* Ω, *and* $\nu\cdot A\tilde{e}=0$ *over* $\partial\Omega$. *For each* $B>0$ *and* $\gamma\geq 1/2$, *consider the following reaction-advection-diffusion equation*

$$\begin{cases} u_t=\nabla\cdot(A(x,y)\nabla u)\;+\;B^{\gamma}\,q(x,y)\cdot\nabla u\;+\;B\,f(x,y,u), & t\in\mathbb{R},\ (x,y)\in\Omega,\\ \nu\cdot A\,\nabla u(t,x,y)=0,\quad t\in\mathbb{R}, & (x,y)\in\partial\Omega.\end{cases}$$

Then

$$\lim_{B\to 0^+}\frac{c^*_{\Omega,A,\,B^{\gamma}q,\,Bf}(e)}{\sqrt{B}}=2\sqrt{⨍_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\,\sqrt{⨍_C\zeta(x,y)dx\,dy},$$

where C *is the periodicity cell of* Ω.

Having the above result one can mark a sample of notes:

The order of B in the denominator of the ratio $c^*_{\Omega,A,\,B^{\gamma}q,\,Bf}(e)/\sqrt{B}$ is independent of γ (it is equal to 1/2). Thus, whenever the advection is null, one gets

$$\lim_{B\to 0^+}\frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}=2\sqrt{⨍_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\,\sqrt{⨍_C\zeta(x,y)dx\,dy}.$$

Therefore, one concludes that the presence of an advection with a factor B^{γ}, where

$\gamma \geq 1/2$, will have no more effect on the limit of the ratio $c^*_{\Omega,A,\,B^\gamma q,\,Bf}(e)/\sqrt{B}$ as the reaction factor B gets very small.

On the other hand, it is easy to check that the assumptions in Theorem 2.5.2 are more general than those in Theorem 2.5.1. Consequently, once we are in the more strict setting, which is that of Theorem 2.5.1, we are able to know both limits of $c^*_{\Omega,A,0,Bf}(e)/\sqrt{B}$ as $B \to +\infty$ and as $B \to 0^+$.

2.6 Variations of the minimal speed with respect to diffusion and reaction factors and with respect to periodicity parameters

After having studied the limits and the asymptotic behaviors of the of the functions $\varepsilon \mapsto c^*_{\Omega,\varepsilon A,0,f}(e)/\sqrt{\varepsilon}$, $M \mapsto c^*_{\Omega,MA,\,M^\gamma q,\,f}(e)/\sqrt{M}$ (for very large M and for $0 \leq \gamma \leq 1/2$), $B \mapsto c^*_{\Omega,A,\,B^\gamma q,\,Bf}(e)/\sqrt{B}$ ($\gamma \geq 1/2$) and $L \mapsto c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$, where L is a periodicity parameter, we move now to investigate the variations of these functions with respect to the diffusion and reaction factors and with respect the periodicity parameter L. The present section will be devoted to discuss and answer these questions.

We sketch first the form of the domain. $\Omega \subseteq \mathbb{R}^N$ is assumed to be in the form $\mathbb{R}\times\omega$ which was taken in section 2.3. As a review, $\Omega = \mathbb{R}\times\omega \subseteq \mathbb{R}^N$, where $\omega \subseteq \mathbb{R}^d\times\mathbb{R}^{N-d-1}$ ($d \geq 0$). If $d = 0$, the subset ω is a bounded open subset of $\mathbb{R}^{N-1}$. While, in the case where $1 \leq d \leq N-1$, ω is a $(L_1,\ldots,L_d)$-periodic open domain of $\mathbb{R}^{N-1}$ which satisfies (3.1.2); and hence, Ω is a $(l,L_1,\ldots,L_d)-$ periodic subset of $\mathbb{R}^N$ that satisfies (3.1.2) with $l > 0$. An element of $\Omega = \mathbb{R}\times\omega$ will be represented as $z = (x,y)$ where $y \in \omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-1-d}$. With a domain of such form, we have:

Theorem 2.6.1 *Let* $e = (1,0,\ldots,0) \in \mathbb{R}^N$. *Assume that* Ω *has the form* $\mathbb{R}\times\omega$ *which is described above, and that the diffusion matrix* $A = A(y)$ *satisfies (2.3.5) together with the assumption*

$$A(x,y)e = A(y)e = \alpha(y)e,\ \textit{for all}\ (x,y) \in \mathbb{R}\times\overline{\omega}; \tag{2.6.1}$$

where $y \mapsto \alpha(y)$ *is a positive* $(L_1,\ldots,L_d)-$ *periodic function defined over* $\overline{\omega}$. *The nonlinearity* f *is assumed to satisfy (2.3.3) and (2.3.4). Moreover, one assumes that, at least, one of* $\tilde{e}\cdot A\tilde{e}$ *and* ζ *is not constant. Besides, the advection field* q *(when it exists) is in the form* $q(x,y) = (q_1(y),0,\ldots,0)$ *where* q_1 *has a zero average over* C, *the* *periodicity cell of ω. For each $\beta > 0$ consider the reaction-advection-diffusion problem*

$$\begin{cases} u_t = \beta\,\nabla\cdot(A(y)\nabla u) \;+\; \sqrt{\beta}\,q_1(y)\,\partial_x u \;+\; f(x,y,u),\; t\,\in\,\mathbb{R},\;(x,y)\,\in\,\mathbb{R}\times\omega, \\ \nu\cdot A\,\nabla u(t,x,y) = 0, \quad t\,\in\,\mathbb{R},\;(x,y)\,\in\,\partial\Omega. \end{cases}$$

Then the map $\beta \mapsto \dfrac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}}$ *is decreasing in $\beta > 0$, and by Theorem 2.4.1, one has*

$$\lim_{\beta\to+\infty}\frac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(y)dy}\,\sqrt{\fint_C \zeta(y)dy},$$

where C is the periodicity cell of ω.

Remark 2.6.2 *In the same setting of Theorem 2.6.1 but with no advection, that is $q_1 \equiv 0$, we still have* $\beta \mapsto \dfrac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}}$ *as a decreasing map in $\beta > 0$. Moreover, if one of the alternatives (2.3.7)-(2.3.8) holds and there is no advection, Theorem 2.3.1 yields that*

$$\lim_{\beta\to 0^+}\frac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}} = 2\sqrt{\max_{y\in\overline{\omega}}\tilde{e}A\tilde{e}(y)}\,\sqrt{\max_{y\in\overline{\omega}}\zeta(y)}.$$

The preceding result yields another one concerned in the variation of the minimal speeds with respect to the periodicity parameter L. In the following, the domain will be the whole space $\mathbb{R}^N$. We choose the diffusion matrix $A(x,y) = A(y)$, the shear flow q and reaction term f to be $(1,\ldots,1)$-periodic and to satisfy some restrictions. For each $L > 0$, we assign the diffusion matrix $A_L(x,y) = A(\dfrac{x}{L},\dfrac{y}{L})$, the advection field $q_L(x,y) = q(\dfrac{x}{L},\dfrac{y}{L})$ and the nonlinearity $f_L = f(\dfrac{x}{L},\dfrac{y}{L},u)$ and we are going to study the variation, with respect to the periodicity parameter L, of the minimal speed $c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$, which corresponds to the reaction-advection-diffusion equation within the $(L,\cdots,L)-$periodic coefficients A_L, q_L and f_L :

Theorem 2.6.3 *Let $e = (1,0,\ldots,0) \in \mathbb{R}^N$. An element $z \in \mathbb{R}^N$ is represented as $z = (x,y) \in \mathbb{R}\times\mathbb{R}^{N-1}$. Assume that $A(x,y) = A(y)$ (for all $(x,y)\in\mathbb{R}^N$) and $f(x,y,u)$ satisfy ((2.3.3), (2.3.4) and 2.3.5) with $\omega = \mathbb{R}^{N-1}$, $d = N-1$, and $l = L_1 = \ldots = L_{N-1} = 1$. Assume furthermore, that for all $y \in \mathbb{R}^{N-1}$, $A(x,y)e = A(y)e = \alpha(y)e$, where $y \mapsto \alpha(y)$ is a positive $(1,\ldots,1)$-periodic function defined over $\mathbb{R}^{N-1}$ and that, at least, one of $\tilde{e}\cdot A\tilde{e}$ and ζ is not constant. Let q be an advection field satisfying (3.1.4) and having the form $q(x,y) = (q_1(y),0\ldots,0)$ for each $(x,y)\in\mathbb{R}^N$. Consider the reaction-advection-diffusion problem,*

$$\begin{array}{c} \forall\,(t,x,y)\in\mathbb{R}\times\mathbb{R}^N, \\ u_t(t,x,y) = \nabla\cdot(A_L(y)\nabla u)(t,x,y) + (q_1)_L(y)\partial_x u(t,x,y) + f_L(x,y,u), \end{array} \tag{2.6.2}$$

whose coefficients are $(L,\ldots,L)-$*periodic with respect to* $(x,y)\in\mathbb{R}^N$.

Then, the map $L\mapsto c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$ *is increasing in* $L>0$.

Remark 2.6.4 *The assumptions of Theorem 2.6.3 can not be fulfilled whenever* $N=1$. *However, assuming that* $N=1$ *and that the function*

$$\frac{\zeta}{<\zeta>_A}+\frac{<a>_H}{a}$$

is not identically equal to 2 *(where* $a(x)$ *is the diffusion factor,* $<a>_H$ *and* $<\zeta>_A$ *are, respectively, the harmonic mean of* $x\mapsto a(x)$ *and arithmetic mean of* $x\mapsto\zeta(x)$ *over* $[0,1]$*), it was proved, in* [13]*, that* $L\mapsto c^*_{\mathbb{R}^N,a_L,q_L,f_L}(e)$ *is increasing in* L *when* L *is close to* 0. *In particular, if* a *is constant and* ζ *is not constant, or if* μ *is constant and* a *is not constant, then* $L\mapsto c^*_{\mathbb{R}^N,a_L,q_L,f_L}(e)$ *is increasing when* L *is close to* 0.

Concerning now the variation with respect to the reaction factor B, we have the following:

Theorem 2.6.5 *Assume that* $\Omega=\mathbb{R}\times\omega$ *and the coefficients* A, q *and* f *satisfy the same assumptions of Theorem 2.6.1. Let* $e=(1,0\ldots,0)$ *and for each* $B>0$, *consider the reaction-advection-diffusion problem*

$$\begin{cases} u_t=\nabla\cdot(A(y)\nabla u)\;+\;\sqrt{B}\,q_1(y)\,\partial_x u\;+\;Bf(x,y,u), & t\in\mathbb{R},\;(x,y)\in\mathbb{R}\times\omega,\\ \nu\cdot A\,\nabla u(t,x,y)=0, \quad t\in\mathbb{R},\;(x,y)\in\partial\Omega. & \end{cases}$$

Then, the map $B\mapsto\dfrac{c^*_{\Omega,A,\sqrt{B}\,q,Bf}(e)}{\sqrt{B}}$ *is increasing in* $B>0$.

As a first note, we mention that Theorem 2.6.5 holds also in the case where there is no advection. On the other hand, Berestycki, Hamel and Nadirashvili [3] proved that the map $B\mapsto c^*_{\Omega,A,q,Bf}(e)$ is increasing in $B>0$ under the assumptions (3.1.2), (3.1.3), (3.1.4), (2.2.4), and (4.1.5) which are less strict than the assumptions considered in our present theorem. However, the present theorem is concerned in the variation of the map $B\mapsto\dfrac{c^*_{\Omega,A,\sqrt{B}\,q,Bf}(e)}{\sqrt{B}}$ rather than that of $B\mapsto c^*_{\Omega,A,q,Bf}(e)$.

Remark 2.6.6 *Owing to the same justifications given after Theorem 2.3.5, one concludes the importance of taking, in section 2.6, an advection in the form of shear flows. To study the variations of the minimal speeds as in Theorems 2.6.1, 2.6.3 and 2.6.5, but in a more general framework (general advection fields, general diffusion, etc...), formula 3.1.18 remains an important tool. However, we will no longer have variational* *formulations as (2.8.65) below. These problems remains open in the general periodic framework.*

2.7 The minimal speed as a function of the positive definite diffusion matrix, Counterexamples

We studied the variation of the function $\beta \mapsto \dfrac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}}$ in the case where $\Omega = \mathbb{R}\times\omega$, q is a shear flow of the form $q(x,y) = (q_1(y),0,\ldots,0)$ on $\overline{\Omega}$, while A and f satisfy (2.3.5), (2.6.1), (2.3.3) and (2.3.4). We obtained that the map $\beta \mapsto \dfrac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}}$ is decreasing with respect to $\beta > 0$ in both cases: $q_1 \not\equiv 0$ or $q_1 \equiv 0$ over ω.

On the other hand, Berestycki, Hamel, and Nadirashvili [3] proved (in part 2 of Theorem 1.10) that: having any periodic domain $\Omega \subseteq \mathbb{R}^N$ satisfying (3.1.2), $q \equiv 0$ and $f = f(u)$, then the map $\beta \mapsto c^*_{\Omega,\beta A,0,f}(e)$ is **increasing** in $\beta > 0$.

Having the two preceding results, there arise naturally the following two questions:

— First: Do we still have the increasing behavior of the minimal speed with respect to the diffusion factor β **in the presence of an advection**, even if the nonlinearity is homogenous?

— Second: Owing to Theorem 3.1.11 (Theorem 1.1 in [3]), the map $D : A \mapsto c^*_{\Omega,A,q,f}(e)$, where A describes the ordered family of positive definite matrices satisfying (3.1.3)(we say that $A = A(x,y) \leq B = B(x,y)$ if and only if for each $(x,y) \in \Omega$ and for each $z \in \mathbb{R}^N$, we have $zA(x,y)z \leq zB(x,y)z$. Also, we say that $A < B$ if and only if for each $(x,y) \in \Omega$ and for each $z \in \mathbb{R}^N$, we have $zA(x,y)z < zB(x,y)z$.) is **well defined** (provided that Ω, q and f satisfy (3.1.2), (3.1.4), (2.2.4) and (4.1.5)). We investigate the variation of the minimal speed of propagation with respect to that of the matrix of diffusion. More precisely, if $A = A(x,y)$ and $B = B(x,y)$ are two positive definite matrices satisfying (3.1.3) and if $A < B$, do we still have $c^*_{\Omega,A,q,f}(e) < c^*_{\Omega,B,q,f}(e)$?

In fact and as it was mentioned above, we have: $\beta \mapsto c^*_{\Omega,\beta A,0,f}(e)$ is **increasing** in $\beta > 0$. In other words, the map D restricted to the sub-family $PD_A = \{\beta A,\ \beta > 0\}$ which is generated by a prefixed matrix A is increasing. So the question becomes now: Does the previous conclusion remain true over the sub-family PD_A in the presence of an advection ?

The answer of the two preceding questions is negative in general. First, we prove that the answer to the second question is negative in general for matrices A and B such that $A \leq B$. We then prove, in section 2.7.2 that, actually, the answer is negative, in general, even when the diffusion matrices A and B are proportional.

2.7.1 A counterexample devoted to answer the second question

Notation 2.7.1 *For each real number b, let A_b denote the $N \times N$ matrix having the form*

$$A_b = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & b & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & b \end{pmatrix}.$$

Proposition 2.7.2 *Let $e = (1, 0, \dots, 0) \in \mathbb{R}^N$, $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$, where ω may or may not be bounded, and let $q = (q_1(y), 0, \dots, 0)$ be a shear flow with a zero average where $q_1 \not\equiv 0$ on $\overline{\omega}$. Assume that the nonlinearity f depend only on y. For each $\varepsilon > 0$, consider the reaction-diffusion-advection problem*

$$\begin{cases} u_t(t,x,y) &= \partial_{xx} u + b\,\Delta_y u + q_1(y)\partial_x u(t,x,y) + f(y,u) \text{ in } \mathbb{R} \times \Omega, \\ &= \nabla \cdot (A_b \nabla u) \;+\; q_1(y)\partial_x u \;+\; f(y,u), \\ \nu_\Omega(x,y) \cdot A_b \nabla_{x,y} u(t,x,y) &= \nu_\omega(y) \cdot \nabla_y u(t,x,y) = 0 \text{ for } (t,x,y) \in \mathbb{R} \times \mathbb{R} \times \partial\omega, \end{cases} \tag{2.7.1}$$

where $\nu_\omega(y)$ denotes the outward unit normal on $\partial\omega$ at the point $y \in \partial\omega$ ($\nu_\Omega(x,y) = (0, \nu_\omega(y))$ is the outward unit normal on $\partial\Omega$ at the point (x,y)) and A_b is the matrix introduced in Notation 2.7.1. Then,

— $$\lim_{b \to +\infty} c^*_{\Omega,\, A_b,\, q,\, f}(e) = 2\sqrt{\fint_C \zeta(y)dy},$$

where C is the periodicity cell of ω.

— *Moreover, if ζ is constant over ω (say $\zeta \equiv \zeta_0$), then*

$$\lim_{b \to 0^+} c^*_{\Omega,\, A_b,\, q,\, f}(e) = \max_{\overline{\omega}}\,(-q_1(y)) + 2\sqrt{\zeta_0}.$$

— *In particular, if $f = f(u)$, then*

$$\begin{aligned} \lim_{b \to +\infty} c^*_{\Omega,\, A_b,\, q,\, f}(e) &= 2\sqrt{f'(0)} \quad \text{and} \\ \lim_{b \to 0^+} c^*_{\Omega,\, A_b,\, q,\, f}(e) &= \max_{\overline{\omega}}\,(-q_1(y)) + 2\sqrt{f'(0)}. \end{aligned}$$

Proof. Consider the following change of variables:

$$\forall\,(t,x,y)\in\mathbb{R}\times\mathbb{R}\times\omega,\quad v(t,x,y)=u(t,\frac{x}{\sqrt{b}},y).$$

One then has: $\forall(t,x,y)\in\mathbb{R}\times\mathbb{R}\times\omega,$

$$v_t(t,x,y)=u_t(t,\frac{x}{\sqrt{b}},y),\;\partial_x v(t,x,y)=\frac{1}{\sqrt{b}}\partial_x\,u(t,\frac{x}{\sqrt{b}},y)$$

$$\partial_{xx}v(t,x,y)=\frac{1}{b}\partial_{x\,x}u(t,\frac{x}{\sqrt{b}},y)\;\text{ and }\;\Delta_y\,v(t,x,y)=\;\Delta_y\,u(t,\frac{x}{\sqrt{b}},y).$$

Owing to the invariance of Ω in the $x-$direction, we have the boundary condition: $\forall\;(t,x,y)\in\mathbb{R}\times\partial\Omega,\quad \nu_{\Omega}(x,y)\cdot\nabla_{x,y}v(t,x,y)=0$. Consequently, the problem (2.7.1) is equivalent to the problem

$$\begin{cases}\forall(t,x,y)\in\mathbb{R}\times\mathbb{R}\times\omega & \\ \qquad v_t(t,x,y) & =b\,\partial_{x\,x}\,v+\,b\Delta_y v\;+\sqrt{b}\,q_1(y)\partial_x v(t,x,y)+f(y,v),\\ & =b\,\Delta_{x,y}v+\;\sqrt{b}\,q_1(y)\partial_x v+\;f(y,v)\text{ in }\mathbb{R}\times\mathbb{R}\times\omega,\\ \nu_{\Omega}(x,y)\cdot\nabla_{x,y}v(t,x,y) & =0\;\text{ for }\;(t,x,y)\in\mathbb{R}\times\mathbb{R}\times\partial\omega.\end{cases}\tag{2.7.2}$$

Let $\alpha^*_{\Omega,\,b\,Id,\,\sqrt{b}\,q,\,f}(e)$ denote the minimal speed of propagation corresponding to problem (2.7.2). Referring to Theorem 2.4.1, and choosing $\gamma=1/2$, one gets

$$\lim_{b\to+\infty}\frac{\alpha^*_{\Omega,\,b\,Id,\,\sqrt{b}\,q,\,f}(e)}{\sqrt{b}}=2\sqrt{\fint_C\zeta(y)dy}.\tag{2.7.3}$$

On the other hand, $\alpha^*_{\Omega,\,b\,Id,\,\sqrt{b}\,q,\,f}(e)=\sqrt{b}\,c^*_{\Omega,\,A_b,\,q,\,f}(e)$. Together with (2.7.3), we obtain that $\lim_{b\to+\infty}c^*_{\Omega,\,A_b,\,q,\,f}(e)=2\sqrt{\fint_C\zeta(y)dy}.$

For the limit as $b\to0^+$, it follows from (2.3.12) in Theorem 2.3.3 that

$$\lim_{b\to0^+}c^*_{\Omega,\,A_b,\,q,\,f}(e)=\lim_{b\to0^+}\frac{\alpha^*_{\Omega,\,b\,Id,\,\sqrt{b}\,q,\,f}(e)}{\sqrt{b}}=\max_{\overline{\omega}}\,(-q_1(y))+2\sqrt{\zeta_0}$$

whenever $\zeta\equiv\zeta_0$ (ζ_0 is a positive constant).

In particular, if $f=f(u)$ is a homogenous KPP nonlinearity, then $\zeta(y)=\zeta_0=f'(0)$ for all $y\in\omega$. $\square$

<u>**Conclusion:**</u> Let $e = (1, 0, \ldots, 0)$ and $\Omega = \mathbb{R} \times \omega$. Choose $f = f(u)$, and $q = (q_1(y), 0, \cdots, 0)$ with $\displaystyle\int_C q_1(y)dy = 0$, and so that there exists $\delta > 0$ satisfying

$$2\sqrt{f'(0)} + \delta < \max_{y \in \overline{\omega}} (-q_1(y)) + 2\sqrt{f'(0)} - \delta.$$

It follows, from Propositions (2.7.2), that there exist $\varepsilon_0 > 0$ and $M_0 > 0$ such that:

$$\forall\, 0 < \varepsilon \leq \varepsilon_0, \quad c^*_{\Omega,\, A_\varepsilon,\, q,\, f}(e) > \max_{y \in \overline{\omega}} (-q_1(y)) + 2\sqrt{f'(0)} - \delta \quad \text{and}$$

$$\forall\, M \geq M_0 > 0, \quad c^*_{\Omega,\, A_M,\, q,\, f}(e) < 2\sqrt{f'(0)} + \delta.$$

Consequently, choosing ε small enough and M large enough, it follows that $A_M \geq A_\varepsilon$ in the sense of the order relation on positive definite matrices; however,

$$c^*_{\Omega,\, A_M,\, q,\, f}(e) < c^*_{\Omega,\, A_\varepsilon,\, q,\, f}(e).$$

Therefore the answer of the second question is negative, in general, even when the non linearity f is homogenous.

2.7.2 A counterexample devoted to answer the first question

In this subsection, we will show an example of a reaction-advection-diffusion problem whose diffusion matrix varies in the sub-family of positive definite matrices $PD_{Id} = \{\beta\, Id,\ \beta > 0\}$, where Id stands for the $N \times N$ identity matrix. In this example, we will apply an advection field which will destruct, even if the nonlinearity f is homogenous, the increasing behavior of the minimal speed with respect to $\beta > 0$ (part (d) of Theorem 2.2.5).

The counterexample

Let $e = (1, 0, \ldots, 0) \in \mathbb{R}^N$, $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$, where ω may or may not be bounded, and let $q = (q_1(y), 0, \ldots, 0)$ be a shear flow with a zero average where $q_1 \not\equiv 0$ on $\overline{\omega}$. Assume that the nonlinearity $f = f(u)$ is a homogenous "KPP" nonlinearity so that

$$0 < 2\sqrt{f'(0)} + \delta < \max_{y \in \overline{\omega}} (-q_1(y)) - \delta, \tag{2.7.4}$$

for some $\delta > 0$.

Step 1. Using Theorem 2.4.1, with $\gamma = 1/2$, we have

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,\, M\, Id,\, \sqrt{M}\, q,\, f}(e)}{\sqrt{M}} = 2\,\sqrt{f'(0)}.$$

Thus, there exists $M_0 := M_0(\delta) > 0$ such that

$$\forall\, M \geq M_0(\delta), \quad 0 < c^*_{\Omega,\, M\, Id,\, \sqrt{M}\, q,\, f}(e) < \sqrt{M}\left(2\sqrt{f'(0)} + \delta\right)$$

Step 2. We fix $M_1 \geq \max(1, M_0(\delta))$. Then,

$$0 < c^*_{\Omega,\, M_1\, Id,\, \sqrt{M_1}\, q,\, f}(e) < \sqrt{M_1}\left(2\sqrt{f'(0)} + \delta\right). \tag{2.7.5}$$

Step 3. For the fixed number M_1, we also have $\sqrt{M_1}\, q$ in the form of shear flows. Theorem 2.3.3 yields that

$$\lim_{\varepsilon\to 0^+} c^*_{\Omega,\, \varepsilon\, Id,\, \sqrt{M_1}\, q,\, f}(e) = \max_{y\in\overline{\omega}}(-\sqrt{M_1}\, q_1(y)) = \sqrt{M_1}\, \max_{y\in\overline{\omega}}(-q_1(y)).$$

Consequently, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that

$$\forall\, \varepsilon \leq \varepsilon_0, \quad c^*_{\Omega,\, \varepsilon\, Id,\, \sqrt{M_1}\, q,\, f}(e) > \sqrt{M_1}\left[\max_{y\in\overline{\omega}}(-q_1(y)) - \delta\right] > 0. \tag{2.7.6}$$

Step 4. Choosing $0 < \varepsilon_1 \ll \min(1, \varepsilon_0)$, and owing to (2.7.4), (2.7.5) and (2.7.6), one then gets

$$c^*_{\Omega,\, \varepsilon_1\, Id,\, \sqrt{M_1}\, q,\, f}(e) > c^*_{\Omega,\, M_1\, Id,\, \sqrt{M_1}\, q,\, f}(e),$$

with $0 < \varepsilon_1 < M_1$.

This shows that the result of part 4 in Theorem 2.2.5 is no longer valid in the presence of an advection field, even if one chooses the nonlinearity f as $f = f(u)$.

Remark 2.7.3 *To meet with the motivation done in the beginning of section 2.7, we mention that there appears two important features in the two counterexamples which were announced in this section. In the counterexample of subsection 2.7.1, the two matrices* $A = A_M$ *and* $B = A_\varepsilon$*, with* M *(resp.* ε*) chosen sufficiently large (resp. sufficiently small), satisfy the properties* $A \geq B$ *and* $c^*_{\Omega,A,q,f}(e) < c^*_{\Omega,B,q,f}(e)$*; however, they are not proportional (that is: there exists no real number* α *such that* $A = \alpha B$*). Meanwhile, in the counterexample of subsection 2.7.2, the matrices* $A = Id$ *and* $B =$

$\varepsilon\, Id$ (ε sufficiently small) are proportional and, in addition, they satisfy: $A > B$ and $c^*_{\Omega,A,q,f}(e) < c^*_{\Omega,B,q,f}(e)$.

2.8 Proofs of the announced results

In this section, we are going to demonstrate the Theorems announced in sections 2.3, 2.4, 2.5, and 2.6. We will proceed in 4 subsections, each devoted to proving the results announced in a corresponding section.

2.8.1 Proofs of Theorems 2.3.1, 2.3.3 and 2.3.4

Proof of Theorem 2.3.1. Under the assumptions of Theorem 2.3.1, we can apply the variational formula (3.1.18) of the minimal speed. Consequently,

$$c^*_{\Omega,\varepsilon A,0,f}(e) \;=\; \min_{\lambda>0} \frac{k_{\Omega,e,\;\varepsilon A,\;0,\,\zeta}(\lambda)}{\lambda}, \tag{2.8.1}$$

where $k_{\Omega,e,\varepsilon A,0,\zeta}(\lambda)$ is the first eigenvalue (for each $\lambda,\ \varepsilon\ >\ 0$) of the eigenvalue problem

$$\begin{cases} L_{\Omega,e,\;\varepsilon A,\;0,\,\zeta,\,\lambda}\,\psi & = k_{\Omega,e,\;\varepsilon A,\;0,\,\zeta}(\lambda)\;\psi(x,y) \text{ over } \mathbb{R}\times\omega;\\ \nu\cdot A\nabla\psi & = 0 \quad \text{on } \mathbb{R}\,\times\,\partial\omega, \end{cases} \tag{2.8.2}$$

and

$$\begin{aligned} L_{\Omega,e,\;\varepsilon A,\;0,\,\zeta,\,\lambda}\psi(x,y) \quad &= \quad \varepsilon\nabla\cdot(A(y)\nabla\psi(x,y)) - 2\,\varepsilon\,\lambda Ae\cdot\nabla\psi(x,y)\;+ \\ &\quad\;\; \left[\varepsilon\,\lambda^2 eA(y)e\; - \lambda\,\varepsilon\nabla\cdot(A(y)e) + \,\zeta(y)\right]\psi(x,y), \end{aligned}$$

for all $(x,y)\in\mathbb{R}\times\omega$.

Initially, the boundary condition in (2.8.2) is $\nu\cdot A\nabla\psi \;=\; \lambda\,\nu\cdot Ae$ on $\partial\Omega = \mathbb{R}\times\partial\omega$; where $\nu(x,y)$ is the unit outward normal at $(x,y)\in\partial\Omega$. However, $\Omega = \mathbb{R}\times\omega$ is invariant in the direction of e which is that of Ae in both alternatives (2.3.7) and (2.3.8). Consequently, $\nu\cdot Ae\equiv 0$ on $\partial\Omega$.

We recall that for all $\lambda>0$, and for all $\varepsilon>0$, we have $k_{\Omega,e,\;\varepsilon A,\;0,\,\zeta}(\lambda)\;>\;0$. Also, the first eigenfunction of (2.8.2) is positive over $\overline{\Omega}\;=\;\mathbb{R}\times\overline{\omega}$, and it is unique up to multiplication by a non zero constant.

In our present setting, whether in (2.3.7) or (2.3.8) and due to the assumption (2.3.4), one concludes that the coefficients in $L_{\Omega,e,\;\varepsilon A,\;0,\,\zeta,\,\lambda}$ are independent of x.

Moreover, in both alternatives (2.3.7) and (2.3.8), the direction of Ae is the same of $e=(1,0,\cdots,0)$. On the other hand, since $\Omega=\mathbb{R}\times\omega$, then for each $(x,y)\in\partial\Omega$, we have $\nu(x,y)=(0;\nu_\omega(y))$, where $\nu_\omega(y)$ is the outward unit normal on $\partial\omega$ at y. Consequently, the first eigenfunction of (2.8.2) is independent of x and **the eigenvalue problem (2.8.2) is reduced to**

$$\begin{cases} L_{\Omega,e,\,\varepsilon A,\,0,\,\zeta,\,\lambda}\phi : & = \varepsilon\,\nabla\cdot(A(y)\nabla\phi(y)) \,+\, [\varepsilon\,\lambda^2 eA(y)e \,+\, \zeta(y)]\,\phi(y) \\ & = k_{\Omega,e,\,\varepsilon A,\,0,\,\zeta}(\lambda)\,\phi \quad \text{over } \omega; \\ \nu(x,y)\cdot A(y)\nabla\phi(y) & = (0;\nu_\omega(y))\cdot A(y)\nabla\phi(y)=0 \quad \text{on } \mathbb{R}\times\partial\omega, \end{cases} \tag{2.8.3}$$

where $\phi=\phi(y)$ is positive over $\overline{\omega}$, $L-$periodic (since the domain ω and the coefficients of $L_{\Omega,e,\,\varepsilon A,\,0,\,\zeta,\,\lambda}$ are $L-$periodic), unique up to multiplication by a constant, and belongs to $C^2(\overline{\omega})$.

In the case where $d\geq 1$, let $C\subseteq\mathbb{R}^{N-1}$ denote the periodicity cell of ω. Otherwise, $d=0$ and one takes $C=\omega$. In both cases, C is bounded. Multiplying the first line of (2.8.3) by ϕ, and integrating by parts over C, one gets

$$-\,k_{\Omega,e,\,\varepsilon A,\,0,\,\zeta}(\lambda) \;=\; \frac{\varepsilon\displaystyle\int_C \nabla\phi\cdot A(y)\nabla\phi\,dy - \int_C\left[\varepsilon\lambda^2 eA(y)e \,+\, \zeta(y)\right]\,\phi^2(y)\,dy}{\displaystyle\int_C \phi^2(y)\,dy}. \tag{2.8.4}$$

One also notes that, in this present setting, the operator $L_{\Omega,e,\,\varepsilon A,0,\zeta,\lambda}$ is self-adjoint and its coefficients are $(L_1,\ldots,L_d)-$periodic with respect $(y_1,\ldots,y_d)$. Consequently, $-\,k_{\Omega,e,\,\varepsilon A,\,0,\,\zeta}(\lambda)$ has the following variational characterization:

$$-\,k_{\Omega,e,\,\varepsilon A,\,0,\,\zeta}(\lambda) = \min_{\varphi\in H^1(C)\backslash\{0\}} \frac{\varepsilon\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi\,dy - \int_C\left[\varepsilon\lambda^2 eA(y)e \,+\, \zeta(y)\right]\,\varphi^2(y)\,dy}{\displaystyle\int_C \varphi^2(y)\,dy}. \tag{2.8.5}$$

In what follows, we will assume that (2.3.7) is the alternative that holds. That is, $eAe=\alpha$ is constant. The proof can be imitated easily whenever we assume that (2.3.8) holds.

The function $y\mapsto\zeta(y)$ is continuous and $(L_1,\ldots,L_d)-$periodic over $\overline{\omega}$, whose periodicity cell C is a bounded subset of $\mathbb{R}^{N-1}$ (whether $d=0$ or $d\geq 1$). Let $y_0\in\overline{C}\subseteq\overline{\omega}$ such that $\max_{y\in\overline{\omega}}\zeta(y)=\zeta(y_0)$ (trivially, this also holds when ζ is constant).

Consequently, we have

$$\forall\, \varphi \in H^1(C) \setminus \{0\},\ \frac{\varepsilon \displaystyle\int_C \nabla\varphi \cdot A\nabla\varphi - \int_C (\varepsilon\alpha\lambda^2 + \zeta(y))\varphi^2}{\displaystyle\int_C \varphi^2(y)\, dy} \geq -\left[\varepsilon\alpha\lambda^2 + \zeta(y_0)\right].$$

This yields that

$$\forall\, \varepsilon > 0,\ \forall\, \lambda > 0,\ -k_{\Omega,e,\ \varepsilon A,\ 0,\ \zeta}(\lambda) \geq -\left[\varepsilon\alpha\lambda^2 + \zeta(y_0)\right]. \tag{2.8.6}$$

Consequently,

$$\forall\, \varepsilon > 0,\ \forall\, \lambda > 0,\ \frac{k_{\Omega,e,\ \varepsilon A,\ 0,\ \zeta}(\lambda)}{\lambda} \leq \lambda\,\alpha\varepsilon + \frac{\zeta(y_0)}{\lambda}. \tag{2.8.7}$$

However, the function $\lambda \mapsto \lambda\alpha\varepsilon + \dfrac{\zeta(y_0)}{\lambda}$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(\varepsilon) = \sqrt{\dfrac{\zeta(y_0)}{\alpha\varepsilon}}$. This minimum is equal to $2\sqrt{\zeta(y_0)} \times \sqrt{\alpha\,\varepsilon}$. From (2.8.7), we conclude that

$$\frac{k_{\Omega,e,\ \varepsilon A,\ 0,\ \zeta}(\lambda(\varepsilon))}{\lambda(\varepsilon)} \leq 2\sqrt{\alpha\varepsilon}\sqrt{\zeta(y_0)}.$$

Finally, (3.1.18) implies that $c^*_{\Omega,\varepsilon A,0,f}(e) = \min\limits_{\lambda>0} \dfrac{k_{\Omega,e,\ \varepsilon A,\ 0,\ \zeta}(\lambda)}{\lambda} \leq 2\sqrt{\alpha\,\varepsilon}\sqrt{\zeta(y_0)}$, or equivalently

$$\forall \varepsilon > 0,\ \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \leq 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{2.8.8}$$

We pass now to prove the other sense of the inequality for $\liminf\limits_{\varepsilon\to 0^+} \dfrac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}}$. We will consider formula (2.8.5), and then organize a suitable function ψ which leads us to a lower bound of $\liminf\limits_{\varepsilon\to 0^+} \dfrac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}}$.

We have $\zeta(y_0) > 0$. Let δ be such that $0 < \delta < \zeta(y_0)$. Thus $0 < \zeta(y_0) - \delta < \max\limits_{\overline{\omega}} \zeta(y)$. The continuity of ζ, over $\overline{C} \subseteq \overline{\omega}$, yields that there exists an open and bounded set $U \subset \overline{C}$ such that

$$\forall\, y \in \overline{U},\ \zeta(y_0) - \delta \leq \zeta(y). \tag{2.8.9}$$

Designate by ψ, a function in $\mathcal{D}(C)$ (a $C^\infty(C)$ function whose support is compact), with $\operatorname{supp}\psi \subseteq \overline{U}$, and $\displaystyle\int_U \psi^2 = 1$. One will have,

$$\begin{aligned}
&\forall\lambda > 0,\ \forall\varepsilon > 0,\\
&-k_{\Omega,e,\ \varepsilon A,\ 0,\zeta}(\lambda) \leq \varepsilon\int_U \nabla\psi\cdot A(y)\nabla\psi\,dy - \int_U \left[\varepsilon\lambda^2 eA(y)e + \zeta(y)\right]\,\psi^2(y)\,dy\\
&\qquad\qquad \leq \varepsilon\int_U \nabla\psi\cdot A(y)\nabla\psi\,dy - \left[\varepsilon\lambda^2\alpha + \zeta(y_0)-\delta\right]\int_U \psi^2(y)\,dy\\
&\qquad\qquad \leq \varepsilon\int_U \alpha_2|\nabla\psi|^2 - \left[\varepsilon\lambda^2\alpha + \zeta(y_0)-\delta\right],\ \text{by (2.3.5)},
\end{aligned}$$

or equivalently

$$\frac{k_{\Omega,e,\ \varepsilon A,\ 0,\zeta}(\lambda)}{\lambda} \geq \lambda\alpha\varepsilon + \frac{1}{\lambda}\,\beta(\varepsilon), \tag{2.8.10}$$

where $\beta(\varepsilon) = \zeta(y_0)-\delta-\varepsilon\displaystyle\int_U \alpha_2|\nabla\psi|^2$. Choosing $0<\varepsilon<\dfrac{\zeta(y_0)-\delta}{\alpha_2\displaystyle\int_U|\nabla\psi|^2}$ (this is possible), we get $\beta(\varepsilon) > 0$.

The map $\lambda\mapsto\lambda\alpha\varepsilon + \dfrac{1}{\lambda}\,\beta(\varepsilon)$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(\varepsilon) = \sqrt{\dfrac{\beta(\varepsilon)}{\varepsilon\alpha}}$. This minimum is equal to $2\sqrt{\varepsilon\,\alpha}\sqrt{\beta(\varepsilon)}$.

Now, referring to formula (2.8.10), one gets

$$\text{For } \varepsilon \text{ small enough,}\quad \frac{k_{\Omega,e,\ \varepsilon A,\ 0,\zeta}(\lambda)}{\lambda} \geq 2\sqrt{\varepsilon\alpha}\,\sqrt{\beta(\varepsilon)}\quad\text{for all } \lambda>0.$$

Together with (3.1.18), we conclude that

$$\text{for } \varepsilon \text{ small enough,}\quad \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \geq 2\sqrt{\beta(\varepsilon)}\sqrt{\alpha}. \tag{2.8.11}$$

Consequently,

$$\begin{aligned}
\liminf_{\varepsilon\to0^+}\frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} &\geq \liminf_{\varepsilon\to0^+} 2\sqrt{\beta(\varepsilon)}\sqrt{\alpha}\\
&= 2\sqrt{\zeta(y_0)-\delta}\sqrt{\alpha}\quad (\text{since } \psi \text{ is independent of } \varepsilon),
\end{aligned}$$

and this holds for all $0<\delta<\zeta(y_0)$. Therefore, one can conclude that

$$\liminf_{\varepsilon\to0^+}\frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \geq 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{2.8.12}$$

Finally, the inequalities (2.8.8) and (2.8.12) imply that $\lim_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}}$ exists, and it is equal to

$$2\sqrt{\alpha}\sqrt{\zeta(y_0)} = 2\sqrt{\max_{\overline{\omega}} eA(y)e}\sqrt{\max_{\overline{\omega}} \zeta(y)}.$$

We note that the same ideas of this proof can be easily applied in the case where the assumption (2.3.8) holds. In (2.3.8), we have ζ is constant; however, eAe is not in general. Meanwhile the converse is true in the case (2.3.7). The little difference is that, in the case of (2.3.8), we choose the subset U (of the proof done above) around the point y_0 where eAe attains its maximum and then we continue by the same way used above. □

Proof of Theorem 2.3.3. We have

$$c^*_{\Omega,\varepsilon A,q,f}(e) \;=\; \min_{\lambda>0} \frac{k_{\Omega,e,\,\varepsilon A,\,q,\,\zeta}(\lambda)}{\lambda}, \tag{2.8.13}$$

where (due to the facts that q is a shear flow, $e = (1, 0, \cdots, 0)$ and e is an eigenvector of the matrix $A(y)$ for all $y \in \overline{\omega}$) $k_{\Omega,e,\,\varepsilon A,\,q,\,\zeta}(\lambda)$ is the principal eigenvalue of the problem

$$\begin{cases} L_{\Omega,e,\,\varepsilon A,\,q,\zeta,\lambda}\psi(x,y) \;=\; k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)\,\psi(x,y) \quad \text{over } \mathbb{R}\times\omega; \\ \nu\cdot A\nabla\psi \;=0 \quad \text{on } \mathbb{R}\,\times\,\partial\omega, \end{cases}$$

with

$$\begin{aligned} L_{\Omega,e,\varepsilon A,\,q,\zeta,\lambda}\,\psi \;&=\; \varepsilon\nabla\cdot(A(y)\nabla\psi) - 2\varepsilon\lambda\,\alpha(y)\,\partial_x\psi + q_1(y)\partial_x\psi \\ &\quad + [\varepsilon\,\lambda^2 eA(y)e \,-\, \lambda q_1(y) + \zeta(y)]\,\psi \text{ over } \mathbb{R}\times\omega. \end{aligned} \tag{2.8.14}$$

The uniqueness of the principal eigenfunction ψ up to multiplication by a constant, yields that one can choose ψ independent of x. Hence, the elliptic operator $L_{\Omega,e,\varepsilon A,\,q,\zeta,\lambda}$ can be reduced to the symmetric operator

$$L_{\Omega,e,\varepsilon A,\,q,\zeta,\lambda}\,\psi = \varepsilon\nabla\cdot(A(y)\nabla\psi) + \left[\varepsilon\,\lambda^2 eA(y)e \,-\, \lambda q_1(y) + \zeta(y)\right]\psi.$$

Consequently,

$$\forall\lambda>0,\,\forall\varepsilon>0,\quad -k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)=\min_{\varphi\in H^1(C)\setminus\{0\}}\frac{\varepsilon\displaystyle\int_C\nabla\varphi\cdot A(y)\nabla\varphi dy+\lambda\int_C q_1(y)\varphi^2-\int_C\left[\lambda^2\varepsilon eA(y)e\,+\,\zeta(y)\right]\,\varphi^2(y)\,dy}{\displaystyle\int_C\varphi^2(y)\,dy}. \tag{2.8.15}$$

Formula (2.8.15) yields that

$$\forall\lambda>0,\,\forall\varepsilon>0,\;-k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)\geq-\lambda\max_{y\in\overline{\omega}}\left(-q_1(y)\right)-\lambda^2\varepsilon\max_{y\in\overline{\omega}}eA(y)e-\max_{y\in\overline{\omega}}\zeta(y),$$

or equivalently

$$\forall\lambda>0,\,\forall\varepsilon>0,\;\frac{k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)}{\lambda}\leq\max_{y\in\overline{\omega}}\left(-q_1(y)\right)+\lambda\varepsilon\max_{y\in\overline{\omega}}eA(y)e+\frac{\max_{y\in\overline{\omega}}\zeta(y)}{\lambda}.$$

Putting $\lambda=\lambda(\varepsilon)=\sqrt{\dfrac{\max_{y\in\overline{\omega}}\zeta(y)}{\varepsilon\max_{y\in\overline{\omega}e\cdot A(y)e}}}>0$ into the last inequality yields that

$$\min_{\lambda>0}\frac{k_{\Omega,e,\,\varepsilon A,\,q,\,\zeta}(\lambda)}{\lambda}\leq\max_{y\in\overline{\omega}}\left(-q_1(y)\right)+2\sqrt{\varepsilon}\sqrt{\max_{y\in\overline{\omega}}e\cdot A(y)e}\sqrt{\max_{y\in\overline{\omega}}\zeta(y)},$$

and hence,

$$\limsup_{\varepsilon\to0^+}c^*_{\Omega,\varepsilon A,q,f}(e)\leq\max_{y\in\overline{\omega}}\left(-q_1(y)\right). \tag{2.8.16}$$

Now, we take $y_0\in C$ (C is the periodicity cell of ω) such that $\max_{y\in\overline{\omega}}\left(-q_1(y)\right)=-q_1(y_0)>0$ (since q is periodic with respect to y, $q_1\not\equiv0$ and q_1 has a zero average) and we take $\delta>0$ such $-q_1(y_0)-\delta>0$. It follows, from the continuity of q_1, that there exists an open subset $U\subset C$ such that $y_0\in U$ and

$$\forall y\in\overline{U},\;-q_1(y)\geq\max_{y\in\overline{\omega}}\left(-q_1(y)\right)-\delta.$$

Let ψ be a function in $\mathcal{D}(C)$ with $\operatorname{supp}\psi\subseteq\overline{U}$, and $\displaystyle\int_U\psi^2=1$. Referring to (2.8.15), it follows that

$$\forall\lambda>0,\forall\varepsilon>0,\quad\frac{k_{\Omega,e,\,\varepsilon A,\,q,\,\zeta}(\lambda)}{\lambda}\;\geq\;-q_1(y_0)-\delta+\lambda\varepsilon\min_{y\in\overline{\omega}}e\cdot Ae\,+\,\frac{1}{\lambda}\,\beta(\varepsilon), \tag{2.8.17}$$

where $\beta(\varepsilon) = \min_{y\in\overline{\omega}} \zeta(y) - \varepsilon \int_U \alpha_2 |\nabla\psi|^2 > 0$ for a small enough $\varepsilon > 0$ ($\alpha_2 > 0$ is the constant appearing in (2.3.5)).

It follows from (2.8.17) that

$$\forall\lambda > 0, \forall\varepsilon > 0, \quad \frac{k_{\Omega,e,\ \varepsilon A,\ q,\zeta}(\lambda)}{\lambda} \ \geq\ -q_1(y_0) - \delta + 2\sqrt{\varepsilon}\sqrt{\min_{y\in\overline{\omega}} e\cdot Ae}\sqrt{\beta(\varepsilon)}.$$

Together with (2.8.13), and since $\delta > 0$ is arbitrary, one gets

$$\liminf_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,q,f}(e) \ \geq\ -q_1(y_0) = \max_{y\in\overline{\omega}}(-q_1(y)). \tag{2.8.18}$$

Finally, (2.8.16) and (2.8.18) complete the proof of (2.3.11).

Similarly, one can use the above technics to prove (2.3.12). However, we will do the proof for the sake of completeness. First, one can easily check that

$$\forall\lambda > 0,\ \forall\varepsilon > 0,\ \frac{k_{\Omega,e,\,\varepsilon A,\sqrt{\varepsilon}\,q,\zeta}(\lambda)}{\lambda\sqrt{\varepsilon}} \leq \max_{y\in\overline{\omega}}\,(-q_1(y)) + \lambda\sqrt{\varepsilon}\max_{y\in\overline{\omega}} eA(y)e + \frac{\max_{y\in\overline{\omega}}\zeta(y)}{\lambda\sqrt{\varepsilon}}.$$

Putting $\lambda = \sqrt{\dfrac{\max_{y\in\overline{\omega}}\zeta(y)}{\max_{y\in\overline{\omega} e\cdot A(y)e}}} > 0$ into the last inequality yields that

$$\min_{\lambda>0} \frac{k_{\Omega,e,\ \varepsilon A,\ q,\zeta}(\lambda)}{\lambda\sqrt{\varepsilon}} \leq \max_{y\in\overline{\omega}}\,(-q_1(y)) + 2\sqrt{\max_{y\in\overline{\omega}} e\cdot A(y)e}\sqrt{\max_{y\in\overline{\omega}}\zeta(y)}.$$

Having eAe and ζ as constants, one then gets

$$\limsup_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,\sqrt{\varepsilon}\,q,f}(e)}{\sqrt{\varepsilon}} \leq \max_{y\in\overline{\omega}}\,(-q_1(y)) + 2\sqrt{\alpha}\sqrt{\zeta_0}. \tag{2.8.19}$$

On the other hand, we take $y_0 \in C$ so that $-q_1(y_0) = \max_{\overline{\omega}}(-q_1(y))$. Also we take any positive number δ so that $0 < \delta_1 < -q_1(y_0)$. It follows, from the continuity of q_1 with respect to y, that there exist three subsets $U \ni y_0$ of C such that

$$-q_1(y) \geq (-q_1(y_0)) - \delta > 0 \text{ for all } y \in \overline{U}.$$

Let ψ be a function in $\mathcal{D}(C)$ so that $\int_C \psi^2 = 1$ and $\psi \equiv 0$ on $C\setminus\overline{U}$. For each $\varepsilon > 0$, we have

$$c^*_{\Omega,\varepsilon A,\sqrt{\varepsilon}\,q,f}(e) \ =\ \min_{\lambda>0} \frac{k_{\Omega,e,\ \varepsilon A,\ \sqrt{\varepsilon}\,q,\,\zeta}(\lambda)}{\lambda},$$

where (owing to the same above justifications)

$$\forall \lambda > 0,\, \forall \varepsilon > 0, \quad -k_{\Omega,e,\,\varepsilon A,\sqrt{\varepsilon}\,q,\zeta}(\lambda) =$$

$$\min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\varepsilon\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi dy + \lambda\int_C q_{_1}(y)\varphi^2 - \int_C \left[\lambda^2\varepsilon eA(y)e \,+\, \zeta(y)\right]\,\varphi^2(y)\,dy}{\displaystyle\int_C \varphi^2(y)\,dy}$$

$$\min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\varepsilon\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi dy + \lambda\int_C q_{_1}(y)\varphi^2 - \int_C \left[\lambda^2\varepsilon\alpha + \zeta_0\right]\varphi^2(y)\,dy}{\displaystyle\int_C \varphi^2(y)\,dy} \tag{2.8.20}$$

since eAe and ζ are constants. Having $\psi\in H^1(C)\setminus\{0\}$, it follows that

$$\forall\lambda>0, \forall\varepsilon>0, \frac{k_{\Omega,e,\,\varepsilon A,\,\sqrt{\varepsilon}q,\,\zeta}(\lambda)}{\lambda} \;\geq\; \sqrt{\varepsilon}(-q_1(y_0)-\delta)+\lambda\varepsilon\alpha+\frac{1}{\lambda}\beta(\varepsilon), \tag{2.8.21}$$

where $\beta(\varepsilon)=\zeta_0-\varepsilon\displaystyle\int_U \alpha_2|\nabla\psi|^2>0$ for $\varepsilon>0$ small enough. Thus,

$$\forall\varepsilon>0,\; c^*_{\Omega,\varepsilon A,\sqrt{\varepsilon}\,q,f}(e)\geq\sqrt{\varepsilon}(-q_1(y_0)-\delta)+2\sqrt{\varepsilon}\sqrt{\alpha}\sqrt{\beta(\varepsilon)}.$$

Since δ was arbitrarily chosen, one the concludes that

$$\liminf_{\varepsilon\to 0^+}\frac{c^*_{\Omega,\varepsilon A,\sqrt{\varepsilon}\,q,f}(e)}{\sqrt{\varepsilon}}\geq -q_1(y_0)+2\sqrt{\alpha}\sqrt{\zeta_0}.$$

Together with (2.8.19), the proof of (2.3.12) is complete. □

Proof of Theorem 2.3.4. Consider the change of variables

$$v(t,x,y)=u(t,Lx,Ly),\quad (t,x,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}.$$

The function u satisfies (2.3.13) if and only if v satisfies

$$v_t(t,x,y)=\frac{1}{L^2}\,\nabla\cdot(A(y)\nabla v)(t,x,y)+f(x,y,v) \text{ over } \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}. \tag{2.8.22}$$

Consequently,

$$\forall\, L>0,\; c^*_{\mathbb{R}^N,\,A_{_L},\,0,\,f_{_L}}(e)\;=\;L\,c^*_{\mathbb{R}^N,\frac{1}{L^2}A,0,f}(e) \tag{2.8.23}$$

Taking $\varepsilon = 1/L^2$, and applying Theorem 2.3.1 to problem (2.8.22), one then has

$$\lim_{L\to+\infty} \frac{c^*_{\mathbb{R}^N,\frac{1}{L^2}A,0,f}(e)}{\sqrt{\dfrac{1}{L^2}}} = \lim_{\varepsilon\to0^+} \frac{c^*_{\mathbb{R}^N,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} = 2\sqrt{\max_{y\in\mathbb{R}^{N-1}}\zeta(y)}\sqrt{\max_{y\in\mathbb{R}^{N-1}} eA(y)e}. \tag{2.8.24}$$

Finally, (2.8.23) together with (2.8.24) complete the proof of Theorem 2.3.4. □

Proof of Theorem 2.3.5. Under the same change of variables considered in the proof of Theorem 2.3.4 above, one gets

$$\begin{aligned} \forall\, L>0,\quad c^*_{\mathbb{R}^N,\,A_L,\,Lq_L,\,f_L}(e) &= L\,c^*_{\mathbb{R}^N,\frac{1}{L^2}A,q,f}(e) \text{ and} \\ c^*_{\mathbb{R}^N,\,A_L,\,q_L,\,f_L}(e) &= L\,c^*_{\mathbb{R}^N,\frac{1}{L^2}A,\frac{1}{L}q,f}(e). \end{aligned}$$

Taking $\varepsilon = \dfrac{1}{L^2}$ and using (2.3.11), then (2.3.15) follows. On the other hand, (2.3.12) implies (2.3.16) whenever eAe and ζ are constant over $\mathbb{R}^{N-1}$. □

2.8.2 Proofs of Theorems 2.4.1 and 2.4.3

Proof of Theorem 2.4.1. The proof will be divided into three steps:

Step 1. According to Theorem 3.1.11, and since $\nu\cdot A\tilde{e} = 0$ on $\partial\Omega$, the minimal speeds $c^*_{\Omega,MA,\,M^\gamma q,\,f}(e)$ are given by:

$$\forall M>0,\ c^*_{\Omega,MA,\,M^\gamma q,\,f}(e) = \min_{\lambda>0}\frac{k_{\Omega,e,\,MA,\,M^\gamma q,\zeta}(\lambda)}{\lambda},$$

where $k_{\Omega,e,\,MA,\,M^\gamma q,\zeta}(\lambda)$ and $\psi^{\lambda,M}$ denote the unique eigenvalue and the positive L-periodic eigenfunction of the problem

$$\begin{aligned} M\nabla\cdot(A\nabla\psi^{\lambda,M}) - 2M\lambda\tilde{e}\cdot A\nabla\psi^{\lambda,M} + M^\gamma q\cdot\nabla\psi^{\lambda,M} + [\lambda^2 M\,\tilde{e}A\tilde{e} - \lambda M^\gamma q\cdot\tilde{e} + \zeta]\psi^{\lambda,M} \\ = k_{\Omega,e,\,MA,\,M^\gamma q,\zeta}(\lambda)\psi^{\lambda,M}\ \text{ in }\Omega, \end{aligned}$$

with $\nu\cdot A\nabla\psi^{\lambda,M} = 0$ on $\partial\Omega$.

For each $\lambda>0$ and $M>0$, let $\lambda' = \lambda\sqrt{M}$, and let $k_{\Omega,e,\,MA,\,M^\gamma q,\zeta}(\lambda) = \mu(\lambda',M)$. Consequently,

$$\forall M>0,\ \frac{c^*_{\Omega,MA,\,M^\gamma q,\,f}(e)}{\sqrt{M}} = \min_{\lambda'>0}\frac{\mu(\lambda',M)}{\lambda'}, \tag{2.8.25}$$

where $\mu(\lambda^{'}, M)$ and $\psi^{\lambda^{'},M}$ are the first eigenvalue and the unique, positive $L-$periodic (with respect to x) eigenfunction of

$$\begin{aligned} & M\nabla\cdot(A\nabla\psi^{\lambda^{'},M})-2\lambda'\sqrt{M}\tilde{e}\cdot A\nabla\psi^{\lambda^{'},M}+M^{\gamma}q\cdot\nabla\psi^{\lambda^{'},M} \\ + & \left[\lambda^{'2}\tilde{e}A\tilde{e}-\frac{\lambda^{'}}{M^{\frac{1}{2}-\gamma}}\,q\cdot\tilde{e}+\zeta\right]\psi^{\lambda^{'},M} \;=\; \mu(\lambda^{'},M)\psi^{\lambda^{'},M} \text{ in } \Omega, \end{aligned} \tag{2.8.26}$$

with $\nu\cdot A\nabla\psi^{\lambda^{'},M}=0$ on $\partial\Omega$.

Owing to the uniqueness, up to multiplication by positive constants, of the first eigenfunction of (2.8.26), one may assume that:

$$\forall\,\lambda^{'}>0,\;\forall\,M>0,\;||\psi^{\lambda^{'},M}||_{L^2(C)}\;=1. \tag{2.8.27}$$

Moreover, for each $M>0,\ \displaystyle\min_{\lambda^{'}>0}\frac{\mu(\lambda^{'},M)}{\lambda^{'}}$ is attained at $\lambda^{'}_M>0$. Thus,

$$\forall M>0,\;\frac{c^{*}_{\Omega,MA,\,M^{\gamma}\,q,\,f}{}^{(e)}}{\sqrt{M}}=\min_{\lambda^{'}>0}\frac{\mu(\lambda^{'},M)}{\lambda^{'}}=\frac{\mu(\lambda^{'}_M,M)}{\lambda^{'}_M}. \tag{2.8.28}$$

The above characterization of $c^{*}_{\Omega,MA,\,M^{\gamma}\,q,\,f}{}^{(e)}/\sqrt{M}$ will be used in the next steps in order to prove that $\displaystyle\liminf_{M\to+\infty} c^{*}_{\Omega,MA,\,M^{\gamma}\,q,\,f}{}^{(e)}/\sqrt{M}$ (resp. $\displaystyle\limsup_{M\to+\infty} c^{*}_{\Omega,MA,\,M^{\gamma}\,q,\,f}{}^{(e)}/\sqrt{M}$) is greater than (resp. less than) $2\sqrt{\displaystyle⨍_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\sqrt{\displaystyle⨍_C \zeta(x,y)dx\,dy}$; and hence, complete the proof.

Step 2. Fix $\lambda^{'}>0$ and $M>0$. We divide (2.8.26) by $\psi^{\lambda^{'},M}$ then, using the facts $\nabla.A\tilde{e}\equiv0$ in Ω and $\nu\cdot A\tilde{e}=0$ on $\partial\Omega$, we integrate by parts over the periodicity cell C. It follows from (3.1.4) and the $L-$periodicity of A, ζ and $\psi^{\lambda^{'},M}$ that

$$\int_C\frac{\nabla\psi^{\lambda^{'},M}\cdot A\nabla\psi^{\lambda^{'},M}}{\left(\psi^{\lambda',M}\right)^2}+\lambda^{'\,2}\int_C\tilde{e}A\tilde{e}+\int_C\zeta=\mu(\lambda^{'},M)|C|, \tag{2.8.29}$$

where $|C|$ denotes the Lebesgue measure of C. Let

$$m_0=⨍_C\tilde{e}A\tilde{e}=\frac{1}{|C|}\int_C\tilde{e}A(x,y)\tilde{e}\,dx\,dy\quad\text{ and }\quad m=⨍_C\zeta(x,y)\,dx\,dy.$$

One concludes that

$$\forall \lambda^{'} > 0,\, \forall M > 0, \quad \mu(\lambda^{'}, M) \;\geq\; {\lambda^{'}}^{2} \fint_{C} \tilde{e}A\tilde{e} + \fint_{C} \zeta = {\lambda^{'}}^{2} m_0 + m,$$

whence

$$\forall \lambda^{'} > 0,\, \forall M > 0, \quad \frac{\mu(\lambda^{'}, M)}{\lambda^{'}} \geq \lambda^{'} m_0 + \frac{m}{\lambda^{'}}. \tag{2.8.30}$$

The right side of (2.8.30) attains its minimum over $\mathbb{R}^{+}$ at $\lambda_0^{'} = \sqrt{\dfrac{m}{m_0}}$. This minimum is equal to $2\sqrt{m_0 m}$.

Consequently, for any $M > 0$, $\dfrac{c^{*}_{\Omega, MA,\, M^{\gamma}\, q,\, f}(e)}{\sqrt{M}} = \min\limits_{\lambda^{'} > 0} \dfrac{\mu(\lambda^{'}, M)}{\lambda^{'}} \geq 2\sqrt{m_0 m}$. This yields that

$$\liminf_{M \to +\infty} \frac{c^{*}_{\Omega, MA,\, M^{\gamma}\, q,\, f}(e)}{\sqrt{M}} \;\geq\; 2\sqrt{\fint_{C} \tilde{e}A\tilde{e}(x,y)dx\,dy}\,\sqrt{\fint_{C} \zeta(x,y)dx\,dy}. \tag{2.8.31}$$

Step 3. Fix $\lambda^{'} > 0$ and $M > 0$. Multiply (2.8.26) by $\psi^{\lambda^{'}, M}$ and integrate by parts over C. Owing to the $L-$periodicity of Ω, A, ζ and $\psi^{\lambda^{'}, M}$, and due to the facts that $\displaystyle\int_{C} \left(\psi^{\lambda^{'}, M}\right)^{2} = 1$, $\nabla \cdot A\tilde{e} \equiv 0$ in Ω, and that $\nu \cdot A\tilde{e} = 0$ on $\partial\Omega$, together with (3.1.4), one gets

$$\begin{aligned} -M \int_{C} \nabla\psi^{\lambda^{'}, M} \cdot A\nabla\psi^{\lambda^{'}, M} + {\lambda^{'}}^{2} \int_{C} \tilde{e}A\tilde{e} \left(\psi^{\lambda^{'}, M}\right)^{2} + \int_{C} \zeta \left(\psi^{\lambda^{'}, M}\right)^{2} \\ - \frac{\lambda^{'}}{M^{\frac{1}{2}-\gamma}} \int_{C} q \cdot \tilde{e} \left(\psi^{\lambda^{'}, M}\right)^{2} \;=\; \mu(\lambda^{'}, M), \end{aligned} \tag{2.8.32}$$

whence

$$\forall \lambda^{'} > 0,\, \forall M > 0,\; 0 < \mu(\lambda^{'}, M) \leq {\lambda^{'}}^{2} \alpha + \beta + \frac{\lambda^{'}}{M^{\frac{1}{2}-\gamma}} \|\, (q \cdot \tilde{e})^{-} \,\|_{\infty},$$

where $\alpha = \max\limits_{(x,y) \in \overline{\Omega}} \tilde{e}A\tilde{e}(x,y)$ and $\beta = \max\limits_{(x,y) \in \overline{\Omega}} \zeta(x,y)$. Together with (2.8.30), one gets

$$\forall \lambda^{'} > 0,\, \forall M > 0,\; 0 < {\lambda^{'}}^{2} m_0 + m \leq \mu(\lambda^{'}, M) \leq {\lambda^{'}}^{2} \alpha + \beta + \frac{\lambda^{'}}{M^{\frac{1}{2}-\gamma}} \|\, (q \cdot \tilde{e})^{-} \,\|_{\infty}. \tag{2.8.33}$$

If $\gamma = \dfrac{1}{2}$, then $\dfrac{\lambda^{'}}{M^{\frac{1}{2}-\gamma}} \|\, (q \cdot \tilde{e})^{-} \,\|_{\infty} = \lambda^{'} \|\, (q \cdot \tilde{e})^{-} \,\|_{\infty}$. On the other hand, if $0 \leq \gamma <$

$\dfrac{1}{2}$, then

$$\frac{\lambda^{'}}{M^{\frac{1}{2}-\gamma}}||\,(q\cdot\tilde{e})^{-}\,||_{\infty}\to 0 \quad \text{as} \quad M\to+\infty.$$

Consequently, the right side of (2.8.33) is bounded above by a positive constant B which does not depend on M and γ. This yields that

$$\forall\lambda^{'}>0, \quad 0<\limsup_{M\to+\infty}\mu(\lambda^{'},M)<+\infty.$$

On the other hand, it follows from (3.1.3) and (2.8.32) that $\forall\lambda^{'}>0, \forall M>0,$

$$\begin{aligned}
0 \ &\leq \alpha_1\int_C|\nabla\psi^{\lambda^{'},M}|^2\leq\int_C\nabla\psi^{\lambda^{'},M}\cdot A\nabla\psi^{\lambda^{'},M}\\
&\leq\frac{1}{M}\left[-\mu(\lambda^{'},M)+\lambda^{'2}\int_C\tilde{e}A\tilde{e}\left(\psi^{\lambda^{'},M}\right)^2+\int_C\zeta\left(\psi^{\lambda^{'},M}\right)^2-\frac{\lambda^{'}}{M^{\frac{1}{2}-\gamma}}\int_C q\cdot\tilde{e}\left(\psi^{\lambda^{'},M}\right)^2\right]\\
&<\frac{B}{M}.
\end{aligned}$$

Meanwhile, $\displaystyle\lim_{M\to+\infty}\frac{B}{M}=0$, one then gets

$$\begin{cases}\forall\lambda^{'}>0, \quad \displaystyle\lim_{M\to+\infty}\int_C|\nabla\psi^{\lambda^{'},M}|^2=0,\\ \forall\lambda^{'}>0,\,\forall M>0, \quad \displaystyle\int_C\left(\psi^{\lambda^{'},M}\right)^2=1.\end{cases} \tag{2.8.34}$$

Fix $\lambda^{'}>0$, and let $(M_n)_n$ be a sequence converging to $+\infty$ as $n\to+\infty$ and such that $\mu(\lambda^{'},M_n)\to l^{\lambda^{'},(M_n)}$ as $n\to+\infty$. It follows, from (2.8.34), that $||\psi^{\lambda^{'},M_n}||_{H^1(C)}\to 1$ as $n\to+\infty$. Thus, the sequence $(\psi^{\lambda^{'},M_n})_n$ is bounded in $H^1(C)$. Therefore, there exists a function $\psi^{\lambda^{'},\infty}\in H^1(C)$ such that, up to extraction of some subsequence, the functions $(\psi^{\lambda^{'},M_n})_n$ converge in $L^2(C)$ strong, $H^1(C)$ weak and almost everywhere in C, to the function $\psi^{\lambda^{'},\infty}$. Consequently, and owing to (2.8.34), $\psi^{\lambda^{'},\infty}$ satisfies

$$\int_C\left(\psi^{\lambda^{'},\infty}\right)^2=1, \text{ and} \tag{2.8.35}$$

$$\left(\int_C|\nabla\psi^{\lambda^{'},\infty}|^2\right)^{\frac{1}{2}}\leq\liminf_{M_n\to+\infty}\left(\int_C|\nabla\psi^{\lambda^{'},M_n}|^2\right)^{\frac{1}{2}}=0. \tag{2.8.36}$$

From (2.8.36), it follows that for all $\lambda^{'}>0$, the function $\psi^{\lambda^{'},\infty}$ is almost everywhere constant over C. On the other hand, the elliptic regularity applied on equation (2.8.26) for $M = M_n$, implies that $\forall\lambda^{'} > 0$, the function $\psi^{\lambda^{'},\infty}$ is continuous over $\overline{C}$. Consequently, referring to (2.8.35), one gets

$$\forall\lambda^{'} > 0, \quad \psi^{\lambda^{'},\infty} = \frac{1}{\sqrt{|C|}} \text{ over } \overline{C}. \tag{2.8.37}$$

Consider now equation (2.8.26). Fix $\lambda^{'}$, take $M = M_n$, and integrate by parts over C. It follows, from (3.1.3), (3.1.4) and the assumptions $\nabla.A\tilde{e} \equiv 0$ over Ω with $\nu.A\tilde{e} = 0$ on $\partial\Omega$, that $\int_C M_n \nabla\cdot(A\nabla\psi^{\lambda^{'},M_n}) = 0$, $\int_C -2\lambda^{'}\sqrt{M_n}\tilde{e}\cdot A\nabla\psi^{\lambda^{'},M_n} = 0$, and $\int_C q\cdot\nabla\psi^{\lambda^{'},M_n} = 0$. Hence,

$$-\frac{\lambda^{'}}{M_n^{\frac{1}{2}-\gamma}}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n} + \lambda^{'2}\int_C \tilde{e}\cdot A\tilde{e}\,\psi^{\lambda^{'},M_n} + \int_C \zeta\,\psi^{\lambda^{'},M_n} = \mu(\lambda^{'},M_n)\int_C \psi^{\lambda^{'},M_n}. \tag{2.8.38}$$

Meanwhile, the functions $\psi^{\lambda^{'},M_n}$ converge to the constant function $\psi^{\lambda^{'},\infty}$ in $L^2(C)$ strong; and hence, in $L^1(C)$ strong (C is bounded, so $L^2(C)$ is embedded in $L^1(C)$). Let $M_n \to +\infty$ in (2.8.38):

In case $\gamma = 1/2$, one has

$$\frac{\lambda^{'}}{M_n^{\frac{1}{2}-\gamma}}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n} = \lambda^{'}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n} \to \lambda^{'}\psi^{\lambda^{'},\infty}\int_C q\cdot\tilde{e} = 0,$$

as $n \to +\infty$ (from (3.1.4)). Also, in the case $0 \leq \gamma < 1/2$, one trivially has

$$\frac{\lambda^{'}}{M_n^{\frac{1}{2}-\gamma}}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n} \to 0 \text{ as } n \to +\infty.$$

Moreover, $\tilde{e}A\tilde{e}$ and ζ are in $L^\infty(C)$. Thus, as $M_n \to +\infty$ in (2.8.38), we get

$$\lambda^{'2}\psi^{\lambda^{'},\infty}\int_C \tilde{e}A\tilde{e} + \psi^{\lambda^{'},\infty}\int_C \zeta = l^{\lambda^{'},(M_n)}\,\psi^{\lambda^{'},\infty}|C|.$$

One concludes that

$$\forall\lambda^{'} > 0, \quad \frac{l^{\lambda^{'},(M_n)}}{\lambda^{'}} = \lambda^{'}\fint_C \tilde{e}A\tilde{e} + \frac{\fint_C \zeta}{\lambda^{'}} = \lambda^{'}m_0 + \frac{m}{\lambda^{'}}. \tag{2.8.39}$$

Whence for $\lambda^{'} = \lambda_0^{'} = \sqrt{\dfrac{m}{m_0}}$, one gets $\dfrac{l^{\lambda_0^{'},(M_n)}}{\lambda_0^{'}} = 2\sqrt{m_0 m}$.

On the other hand, for all M_n,

$$\frac{c^*_{\Omega, M_nA,\, M_n^\gamma\, q,\, f}(e)}{\sqrt{M_n}} = \inf_{\lambda'>0} \frac{\mu(\lambda^{'}, M_n)}{\lambda^{'}} \leq \frac{\mu(\lambda_0^{'}, M_n)}{\lambda_0^{'}}. \tag{2.8.40}$$

Passing $M_n \to +\infty$, one gets $\limsup_{M_n \to +\infty} \dfrac{c^*_{\Omega, M_nA,\, M_n^\gamma\, q,\, f}(e)}{\sqrt{M_n}} \leq \dfrac{l^{\lambda_0^{'},(M_n)}}{\lambda_0^{'}} = 2\sqrt{m_0 m}$, and this holds for all sequences $\{M_n\}_n$ converging to $+\infty$. Thus,

$$\limsup_{M \to +\infty} \frac{c^*_{\Omega, MA,\, M^\gamma\, q,\, f}(e)}{\sqrt{M}} \leq\; 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\, dxdy}\, \sqrt{\fint_C \zeta(x,y)\, dxdy}. \tag{2.8.41}$$

Having (2.8.31) together with (2.8.41), the proof of Theorem 2.4.1 is complete. □

Proof of Theorem 2.4.3. We will consider the change of variables similar to that made in the proof of Theorem 2.3.4:

$$v(t,x,y) = u(t, Lx, Ly), \quad (t,x,y) \in \mathbb{R} \times \mathbb{R}^N.$$

After the same calculations done there, one gets that u satisfies (2.4.1) if and only if v satisfies

$$v_t(t,x,y) = \frac{1}{L^2} \nabla \cdot (A(x,y)\nabla v)(t,x,y) + \frac{1}{L}\, q \cdot \nabla v(t,x,y) + f(x,y,v) \text{ over } \mathbb{R} \times \mathbb{R}^N. \tag{2.8.42}$$

Consequently,

$$\forall\, L > 0,\; c^*_{\mathbb{R}^N,\, A_L,\, q_L,\, f_L}(e) \;=\; L\, c^*_{\mathbb{R}^N, \frac{1}{L^2}A, \frac{1}{L}q, f}(e). \tag{2.8.43}$$

On the other hand, the coefficients and the domain of problem (2.8.42) satisfy all the assumptions of Theorem 2.4.1. Taking $M = 1/L^2$ and $\gamma = 1/2$, then (2.8.42) can be rewritten as

$$v_t(t,x,y) = M\, \nabla \cdot (A(x,y)\nabla v)(t,x,y) + M^{\frac{1}{2}}\, q \cdot \nabla v(t,x,y) + f(x,y,v) \text{ over } \mathbb{R} \times \mathbb{R}^N.$$

In this situation, the periodicity cell of the whole space $\mathbb{R}^N$ is $C = [0,1] \times \cdots \times [0,1]$.

It follows, from Theorem 2.4.1, that

$$\begin{aligned}\lim_{L\to 0^+} \frac{c^*_{\mathbb{R}^N,\frac{1}{L^2}A,\frac{1}{L}q,f}(e)}{\sqrt{\dfrac{1}{L^2}}} &= \lim_{M\to+\infty} \frac{c^*_{\mathbb{R}^N,M\,A,M^{\frac{1}{2}}\,q,f}(e)}{\sqrt{M}} \\ &= 2\sqrt{⨍_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\sqrt{⨍_C \zeta(x,y)dx\,dy}.\end{aligned} \tag{2.8.44}$$

Having (2.8.43) together with (2.8.44), the proof of Theorem 2.4.3 is complete. □

2.8.3 Proofs of Theorems 2.5.1 and 2.5.2

Proof of Theorem 2.5.1. The main ideas of this proof are similar to those in the demonstration of Theorem 2.3.1. Applying the variational formula (3.1.18) of the minimal speed, one gets

$$c^*_{\Omega,A,0,\,Bf}(e) = \min_{\lambda>0} \frac{k_{\Omega,e,\,A,\,0,\,B\zeta}(\lambda)}{\lambda}, \tag{2.8.45}$$

where $k_{\Omega,e,A,0,B\zeta}(\lambda)$ is the first eigenvalue (for each λ, $B>0$) of the eigenvalue problem:

$$\begin{cases} L_{\Omega,e,\,A,\,0,B\zeta,\lambda}\;\psi(x,y) &= k_{\Omega,e,\,A,\,0,B\zeta}(\lambda)\;\psi(x,y) \text{ over } \mathbb{R}\times\omega; \\ \nu\cdot A\nabla\psi &= 0 \quad \text{on } \mathbb{R}\times\partial\omega, \end{cases} \tag{2.8.46}$$

and

$$\begin{aligned} L_{\Omega,e,\,A,\,0,B\,\zeta,\lambda}\psi(x,y) \;=\; & \nabla\cdot(A(y)\nabla\psi(x,y)) - 2\,\lambda Ae\cdot\nabla\psi(x,y) + \\ & \left[\lambda^2 eA(y)e - \lambda\,\nabla\cdot(A(y)e) + B\zeta(y)\right]\psi(x,y), \end{aligned}$$

for each $(x,y)\in\mathbb{R}\times\omega$.

We recall that for all $\lambda>0$, and for all $B>0$, we have $k_{\Omega,e,\,A,\,0,\,B\zeta}(\lambda)>0$. Also, the first eigenfunction of (2.8.46) is positive over $\overline{\Omega}=\mathbb{R}\times\overline{\omega}$, and it is unique up to multiplication by a non zero constant.

Moreover, whether in (2.3.7) or (2.3.8) and due to (2.3.4), one concludes that the coefficients in $L_{\Omega,e,\,A,\,0,\,B\zeta,\,\lambda}$ are independent of x. Hence, the first eigenfunction of

(2.8.46) is independent of x and the eigenvalue problem (2.8.46) is reduced to

$$\left\{\begin{array}{rl} L_{\Omega,e,A,0,\,B\zeta,\,\lambda}\phi & := \nabla\cdot(A(y)\nabla\phi(y)) + \left[\lambda^2 eA(y)e + B\zeta(y)\right]\phi(y) \\ & = k_{\Omega,e,A,0,B\zeta}(\lambda)\,\phi \quad \text{over } \omega; \\ \nu(x,y)\cdot A(y)\nabla\phi(y) & = (0;\nu_\omega(y))\cdot A(y)\nabla\phi(y) = 0 \quad \text{on } \mathbb{R}\times\partial\omega, \end{array}\right. \tag{2.8.47}$$

where $\phi = \phi(y)$ is positive over $\overline{\omega}$, $L-$periodic (since the domain ω and the coefficients of $L_{\Omega,e,A,0,\,B\zeta,\,\lambda}$ are $L-$periodic), unique up to multiplication by a constant, and belongs to $C^2(\overline{\omega})$.

In the case where $d \geq 1$, let $C \subseteq \mathbb{R}^{N-1}$ denote the periodicity cell of ω. Otherwise, $d = 0$ and one takes $C = \omega$. In both cases, C is bounded. Multiplying the first line of (2.8.47) by ϕ, and integrating by parts over C, one gets

$$-\,k_{\Omega,e,A,0,\,B\zeta}(\lambda) = \frac{\displaystyle\int_C \nabla\phi\cdot A(y)\nabla\phi\,dy - \int_C \left[\lambda^2 eA(y)e + B\,\zeta(y)\right]\,\phi^2(y)\,dy}{\displaystyle\int_C \phi^2(y)\,dy}. \tag{2.8.48}$$

One also notes that, in this present setting, the operator $L_{\Omega,e,A,0,\,B\zeta,\,\lambda}$ is self-adjoint and its coefficients are $(L_1,\ldots,L_d)-$periodic with respect $(y_1,\ldots,y_d)$. Consequently, $-\,k_{\Omega,e,A,0,\,B\zeta}(\lambda)$ has the following variational characterization:

$$-\,k_{\Omega,e,A,0,\,B\zeta}(\lambda) = \min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi\,dy - \int_C \left[\lambda^2 eA(y)e + B\,\zeta(y)\right]\,\varphi^2(y)\,dy}{\displaystyle\int_C \varphi^2(y)\,dy}. \tag{2.8.49}$$

In what follows, we will assume that (2.3.7) is the alternative that holds. That is, $eAe = \alpha$ is constant. The proof can be imitated easily whenever we assume that (2.3.8) holds.

The function $y \mapsto \zeta(y)$ is continuous and $(L_1,\ldots,L_d)-$periodic over $\overline{\omega}$, whose periodicity cell C is a bounded subset of $\mathbb{R}^{N-1}$ (whether $d=0$ or $d\geq 1$). Let $y_0 \in \overline{C} \subseteq \overline{\omega}$ such that $\max_{y\in\overline{\omega}} \zeta(y) = \zeta(y_0)$ (trivially, this also holds when ζ is constant). Consequently, we have

$$\forall\,\varphi\in H^1(C)\setminus\{0\},\; \frac{\displaystyle\int_C \nabla\varphi\cdot A\nabla\varphi - \int_C (\alpha\lambda^2 + B\,\zeta(y))\varphi^2}{\displaystyle\int_C \varphi^2(y)\,dy} \geq -\left[\alpha\lambda^2 + B\,\zeta(y_0)\right].$$

This yields that

$$\forall\, B > 0,\, \forall\, \lambda > 0,\, -k_{\Omega,e,\,A,\,0,\,B\,\zeta}(\lambda) \;\geq\; -\left[\alpha\lambda^2 \,+\, B\,\zeta(y_0)\right]. \tag{2.8.50}$$

Consequently,

$$\forall\, B > 0,\, \forall\, \lambda > 0,\, \frac{k_{\Omega,e,\,A,\,0,\,B\zeta}(\lambda)}{\lambda} \;\leq\; \lambda\,\alpha \,+\, \frac{B\,\zeta(y_0)}{\lambda}. \tag{2.8.51}$$

However, the function $\lambda \mapsto \lambda\alpha + (B\,\zeta(y_0)/\lambda)$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(B) \;=\; \sqrt{\dfrac{B\,\zeta(y_0)}{\alpha}}$. This minimum is equal to $2\sqrt{B\zeta(y_0)} \times \sqrt{\alpha}$.

From (2.8.51), we conclude that: $\dfrac{k_{\Omega,e,\,A,\,0,\,B\zeta}(\lambda(B))}{\lambda(B)} \;\leq\; 2\sqrt{B\,\alpha}\sqrt{\zeta(y_0)}$.

Finally, (3.1.18) implies that

$$c^*_{\Omega,A,0,\,Bf}(e) \;=\; \min_{\lambda>0} \frac{k_{\Omega,e,\,A,\,0,\,B\zeta}(\lambda)}{\lambda} \;\leq\; 2\sqrt{B\,\alpha}\sqrt{\zeta(y_0)},$$

or equivalently

$$\forall\, B \,>0,\, \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} \;\leq\; 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{2.8.52}$$

We pass now to prove the other sense of the inequality for $\liminf\limits_{B\to+\infty} \dfrac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}$. We will consider formula (2.8.5), and then organize a suitable function ψ which leads us to a lower bound of $\liminf\limits_{B\to+\infty} \dfrac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}$.

We have $\zeta(y_0) \;>\; 0$. Let δ be such that $0 < \delta < \zeta(y_0)$. Thus $0 < \zeta(y_0) - \delta < \max\limits_{\overline{\omega}} \zeta(y)$. The continuity of ζ, over $\overline{C} \subseteq \overline{\omega}$, yields that there exists an open and bounded set $U \subset \overline{C}$ such that

$$\zeta(y_0) - \delta \leq \zeta(y),\ \forall\, y \in \overline{U}. \tag{2.8.53}$$

Designate by ψ, a function in $\mathcal{D}(C)$ (a $C^\infty(C)$ function whose support is compact), with $\operatorname{supp}\psi \subseteq \overline{U}$, and $\displaystyle\int_U \psi^2 = 1$. One will have,

$$
\begin{aligned}
&\forall\lambda > 0,\ \forall\, B > 0,\\
-\,k_{\Omega,e,\,A,\,0,\,B\,\zeta}(\lambda) \;&\leq \int_U \nabla\psi\cdot A(y)\nabla\psi\,dy - \int_U \left[\lambda^2 eA(y)e \,+\, B\,\zeta(y)\right]\,\psi^2(y)\,dy\\
&\leq \int_U \nabla\psi\cdot A(y)\nabla\psi\,dy - \left[\lambda^2\alpha \,+\, B\,(\zeta(y_0)-\delta)\right] \text{ (by (2.8.53))}\\
&\leq \int_U \alpha_2|\nabla\psi|^2 \,-\, \left[\lambda^2\alpha \,+\, B\,\left(\zeta(y_0)-\delta\right)\right] \text{ by (2.3.5)},
\end{aligned}
$$

or equivalently

$$\frac{k_{\Omega,e,\,A},\ 0,B\,\zeta(\lambda)}{\lambda} \;\geq\; \lambda\alpha \,+\, \frac{B}{\lambda}\,\rho(B), \tag{2.8.54}$$

where $\rho(B) = \zeta(y_0) - \delta - \dfrac{1}{B}\displaystyle\int_U \alpha_2|\nabla\psi|^2$. Choosing B large enough, we get $\rho(B)\,>0$ (this is possible since $\zeta(y_0)-\delta\,>\,0$ and also $\displaystyle\int_U \alpha_2|\nabla\psi|^2\,>\,0$). The map $\lambda\mapsto \lambda\alpha\,+\,\dfrac{B}{\lambda}\,\rho(B)$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(\varepsilon) = \sqrt{\dfrac{B\,\rho(B)}{\alpha}}$. This minimum is equal to $2\sqrt{B\,\alpha}\,\sqrt{\rho(B)}$.

Now, referring to formula (2.8.54), one gets:

$$\text{for } B \text{ large enough, } \frac{k_{\Omega,e,\,A,\,0,\,B\zeta}(\lambda)}{\lambda} \;\geq\; 2\sqrt{B\,\alpha}\,\sqrt{\rho(B)} \quad \text{for all } \lambda>0.$$

Together with (3.1.18), we conclude that

$$\text{for } B \text{ large enough, } \frac{c^*_{\Omega,A,0,\,Bf}(e)}{\sqrt{B}} \;\geq\; 2\sqrt{\rho(B)}\sqrt{\alpha}. \tag{2.8.55}$$

Consequently,

$$
\begin{aligned}
\liminf_{B\to+\infty}\frac{c^*_{\Omega,A,0,\,Bf}(e)}{\sqrt{B}} \;&\geq\; \liminf_{B\to+\infty} 2\sqrt{\rho(B)}\sqrt{\alpha}\\
&=\; 2\sqrt{\zeta(y_0)-\delta}\sqrt{\alpha} \quad \text{(since } \psi \text{ is independent of } B\text{)},
\end{aligned}
$$

and this holds for all $0<\delta<\zeta(y_0)$. Therefore, one can conclude that

$$\liminf_{B\to+\infty}\frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} \;\geq\; 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{2.8.56}$$

Finally, the inequalities (2.8.52) and (2.8.56) imply that $\lim\limits_{B\to+\infty}\dfrac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}$ exists, and it is equal to $2\sqrt{\alpha}\sqrt{\zeta(y_0)} = 2\sqrt{\max\limits_{\overline{\omega}} eA(y)e}\sqrt{\max\limits_{\overline{\omega}}\zeta(y)}$.

The above proof was done while assuming that the alternative (2.3.7) holds. The same ideas of this proof can be easily applied in the case where alternative (2.3.8) holds. In (2.3.8), we have ζ is constant; however, eAe is not in general. Meanwhile the converse is true in the case (2.3.7). The little difference is that, in the case of (2.3.8), we chsose the subset U (of the proof done above) around the point y_0 where eAe attains its maximum and then we continue by the same way used above. □

Proof of Theorem 2.5.2. According to Theorem 3.1.11, and since $\nu\cdot A\tilde{e}=0$ on $\partial\Omega$, the minimal speeds $c^*_{\Omega,A,\,B^\gamma q,\,Bf}(e)$ are given by:

$$\forall B>0,\ c^*_{\Omega,A,\,B^\gamma q,\,Bf}(e)=\min_{\lambda>0}\frac{k_{\Omega,e,A,\,B^\gamma q,\,B\zeta}(\lambda)}{\lambda},$$

where $k_{\Omega,e,A,\,B^\gamma q,\,B\zeta}(\lambda)$ and $\psi^{\lambda,B}$ denote the unique eigenvalue and the positive L-periodic eigenfunction of the problem

$$\begin{aligned}\nabla\cdot(A\nabla\psi^{\lambda,B})-2\lambda\tilde{e}\cdot A\nabla\psi^{\lambda,B}+B^\gamma q\cdot\nabla\psi^{\lambda,B}+\left[\lambda^2\,\tilde{e}A\tilde{e}-\lambda B^\gamma q\cdot\tilde{e}+B\,\zeta\right]\psi^{\lambda,B}\\ =k_{\Omega,e,A,\,B^\gamma q,\,B\zeta}(\lambda)\,\psi^{\lambda,B}\ \text{ in }\Omega,\text{ with }\nu\cdot A\nabla\psi=\nu\cdot A\nabla\psi^{\lambda,B}=0\text{ on }\partial\Omega.\end{aligned}$$

For each $\lambda>0$ and $B>0$, let $\lambda'=\lambda/\sqrt{B}$, and let $k_{\Omega,e,A,\,B^\gamma q,\,B\zeta}(\lambda)=\mu(\lambda',B)$. Consequently,

$$\forall B>0,\ \frac{c^*_{\Omega,A,\,B^\gamma q,\,Bf}(e)}{\sqrt{B}}=\min_{\lambda'>0}\frac{\mu(\lambda',B)}{\lambda' B},\tag{2.8.57}$$

where $\mu(\lambda',B)$ and $\psi^{\lambda',B}$ are the first eigenvalue and the unique, positive $L-$periodic (with respect to x) eigenfunction of

$$\begin{aligned}&\nabla\cdot(A\nabla\psi^{\lambda',B})-2\lambda'\sqrt{B}\tilde{e}\cdot A\nabla\psi^{\lambda',B}+B^\gamma q\cdot\nabla\psi^{\lambda',B}\\ &+\left[\lambda'^2B\,\tilde{e}A\tilde{e}-\lambda' B^{\gamma+\frac12}\,q\cdot\tilde{e}+B\zeta\right]\psi^{\lambda',B}\ =\ \mu(\lambda',B)\psi^{\lambda',B}\text{ in }\Omega,\end{aligned}\tag{2.8.58}$$

with $\nu\cdot A\nabla\psi^{\lambda',B}=0$ on $\partial\Omega$.

Owing to the uniqueness, up to multiplication by positive constants, of the first eigenfunction of (2.8.58), one may assume that:

$$\forall\,\lambda'>0,\ \forall\,B>0,\ ||\psi^{\lambda',B}||_{L^2(C)}=1.\tag{2.8.59}$$

Moreover, for each $B > 0$, $\min\limits_{\lambda' > 0} \dfrac{\mu(\lambda', B)}{\lambda' B}$ is attained at $\lambda'_B > 0$. Thus,

$$\forall B > 0, \ \frac{c^*_{\Omega, A, \, B^\gamma q, \, Bf}(e)}{\sqrt{B}} = \min_{\lambda' > 0} \frac{\mu(\lambda', B)}{\lambda' B} = \frac{\mu(\lambda'_B, B)}{B \, \lambda'_B}. \tag{2.8.60}$$

Having the above characterization, one can now imitate the steps 2 and 3 in the proof of Theorem 2.4.1 to prove that

$$\liminf_{B \to 0^+} c^*_{\Omega, A, \, B^\gamma q, \, Bf}(e)/\sqrt{B}$$

(resp. $\limsup\limits_{B \to 0^+} c^*_{\Omega, A, \, B^\gamma q, \, Bf}(e)/\sqrt{B}$) is greater than (resp. less than)

$$2\sqrt{\fint_C \tilde{e} A \tilde{e}(x,y) dx \, dy} \, \sqrt{\fint_C \zeta(x,y) dx \, dy};$$

and hence, complete the proof of Theorem 2.5.2. □

2.8.4 Proofs of Theorems 2.6.1, 2.6.3, and 2.6.5

Proof of Theorem 2.6.1. Referring to Theorem 3.1.11, it follows that for each $\beta > 0$, we have:

$$\frac{c^*_{\Omega, \beta A, \sqrt{\beta}\, q, f}(e)}{\sqrt{\beta}} = \min_{\lambda > 0} \frac{k_{\Omega, e, \, \beta A, \, \sqrt{\beta}\, q, \zeta}(\lambda)}{\lambda \sqrt{\beta}},$$

where $k_{\Omega, e, \, \beta A, \, \sqrt{\beta}\, q, \zeta}(\lambda)$ is the first eigenvalue of the problem

$$\begin{cases} L_{\Omega, e, \, \beta A, \, \sqrt{\beta}\, q, \zeta, \lambda} \psi(x,y) &= k_{\Omega, e, \, \beta A, \, \sqrt{\beta}\, q, \zeta}(\lambda) \, \psi(x,y) \text{ over } \mathbb{R} \times \omega; \\ \nu . A \nabla \psi &= 0 \text{ on } \mathbb{R} \times \partial \omega, \end{cases} \tag{2.8.61}$$

where

$$\begin{aligned} L_{\Omega, e, \beta A, \sqrt{\beta}\, q, \zeta, \lambda} \, \psi &= \beta \nabla \cdot (A(y) \nabla \psi) - 2\beta\lambda \, \alpha(y) \, \partial_x \psi + \sqrt{\beta}\, q_1(y) \partial_x \psi \\ &\quad + \Big[\beta \, \lambda^2 e A(y) e - \lambda \sqrt{\beta}\, q_1(y) + \zeta(y) \Big] \, \psi \text{ over } \mathbb{R} \times \omega. \end{aligned}$$

The boundary condition follows so from the facts that $\Omega = \mathbb{R} \times \omega$, $e = (1, 0, \ldots, 0)$ and that $A(y)e = \alpha(y)e$ over ω. These yield that $\nu \cdot Ae = 0$ over $\partial\Omega$ and $\nabla \cdot Ae = 0$. Moreover, for each $(x,y) \in \partial\Omega$, we have $\nu(x,y) = (0; \nu_\omega(y))$, where $\nu_\omega(y)$ is the outward unit normal on $\partial\omega$ at y.

On the other hand, the function ψ is positive , $(L_1,\ldots,L_d)-$periodic with respect to y, and unique up to multiplication by non-zero constants. Meanwhile, the coefficients A, q and ζ are independent of x. Thus the eigenfunction ψ will be independent of x and our eigenvalue problem is reduced to

$$\begin{cases} \beta\nabla\cdot(A(y)\nabla\psi(y)) + \left[\beta\,\lambda^2 eA(y)e\, - \lambda\sqrt{\beta}q_1(y) + \zeta(y)\right]\psi(y) \\ \qquad = k_{\Omega,e,\,\beta A,\,\sqrt{\beta}q,\zeta}(\lambda)\psi(y) \text{ for all } y\in\omega, \\ \nu(x,y)\cdot A(y)\nabla\psi(y) \;=\; (0;\nu_\omega(y))\cdot A(y)\nabla\psi(y) \;=\; 0 \text{ on } \mathbb{R}\times\partial\omega. \end{cases} \tag{2.8.62}$$

For each $\lambda>0$ and $\beta>0$, let $\lambda^{'}=\lambda\sqrt{\beta}$, and let $k_{\Omega,e,\,\beta A,\,\sqrt{\beta}\,q,\zeta}(\lambda)=\mu(\lambda^{'},\beta)$. Since for each $\beta>0$, $\displaystyle\min_{\lambda>0}\frac{k_{\Omega,e,\,\beta A,\,\sqrt{\beta}\,q,\zeta}(\lambda)}{\lambda}$ is attained at $\lambda(\beta)$, it follows that

$$\forall\,\beta>0,\qquad \frac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}} = \min_{\lambda^{'}>0}\frac{\mu(\lambda^{'},\beta)}{\lambda^{'}}, \tag{2.8.63}$$

where $\mu(\lambda^{'},\beta)$ is the first eigenvalue of the problem:

$$\begin{cases} L^{\beta}_{\lambda^{'}}\psi &= \beta\nabla\cdot(A(y)\nabla\psi) + \left[\lambda^{'2}eA(y)e - \lambda^{'}q_1(y)+\zeta(y)\right]\psi = \mu(\lambda^{'},\beta)\psi \text{ in } \omega, \\ \nu\cdot A\nabla\psi &= 0 \text{ on } \partial\omega. \end{cases} \tag{2.8.64}$$

The elliptic operator $L^{\beta}_{\lambda^{'}}$ in (2.8.64) is self-adjoint. Consequently, the first eigenvalue $\mu(\lambda^{'},\beta)$ has the following characterization: [3]

$$\forall\lambda^{'}>0,\,\forall\beta>0,\quad -\mu(\lambda^{'},\beta) =$$

$$\min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\beta\displaystyle\int_C\nabla\varphi\cdot A(y)\nabla\varphi dy + \lambda^{'}\int_C q_1(y)\varphi^2 - \int_C\left[\lambda^{'2}eA(y)e+\zeta(y)\right]\varphi^2(y)dy}{\displaystyle\int_C\varphi^2(y)dy} \tag{2.8.65}$$

$$= \min_{\varphi\in H^1(C)\setminus\{0\}} R(\lambda^{'},\beta,\varphi).$$

For each $\lambda^{'}$ and $\beta>0$, $\varphi\mapsto R(\lambda^{'},\beta,\varphi)$ attains its minimum over $H^1(C)\setminus\{0\}$ at $\psi^{\lambda^{'},\beta}$, the eigenfunction of the problem (2.8.64). On the other hand, $\beta\mapsto R(\lambda^{'},\beta,\varphi)$ is increasing as an affine function in β. Consequently, fixing $\lambda^{'}>0$ and taking $\beta>$

3. To have an idea, multiply (2.8.64) by the positive, $(L_1,\ldots,L_d)-$periodic function ψ and integrate by parts over the periodicity cell C of the the domain ω.

$\beta' > 0$:

$$\begin{aligned} -\mu(\lambda', \beta) &= R(\lambda', \beta, \psi^{\lambda',\beta}) > R(\lambda', \beta', \psi^{\lambda',\beta}) \\ &\geq \min_{\varphi \in H^1(C)\setminus\{0\}} R(\lambda', \beta', \varphi) = -\mu(\lambda', \beta'). \end{aligned} \tag{2.8.66}$$

In other words, for all $\lambda' > 0$, the function $\beta \mapsto \mu(\lambda', \beta)$ is decreasing. Concerning now the function $\beta \mapsto c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)/\sqrt{\beta}$, one takes randomly $\beta > \beta' > 0$, hence

$$\begin{aligned} \frac{c^*_{\Omega,\beta' A,\sqrt{\beta'}\,q,f}(e)}{\sqrt{\beta'}} &= \frac{\mu(\lambda'(\beta'),\beta')}{\lambda'(\beta')} > \frac{\mu(\lambda'(\beta'),\beta)}{\lambda'(\beta')} \\ &\geq \min_{\lambda'>0} \frac{\mu(\lambda',\beta)}{\lambda'} = \frac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}}, \end{aligned}$$

which means that the function $\beta \mapsto c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)/\sqrt{\beta}$ is decreasing.

Finally, when $\beta \to +\infty$, one can easily check that the hypothesis of Theorem 2.4.1 are satisfied; hence, one has the limit at $+\infty$, and that completes the proof of Theorem 2.6.1. □

Proof of Theorem 2.6.3. Consider the change of variables $v(t,x,y) = u(t, Lx, Ly)$, for any $(t,x,y) \in \mathbb{R} \times \mathbb{R}^N$. One consequently has,

$$\forall\, L > 0,\; c^*_{\mathbb{R}^N, A_L, q_L, f_L}(e) \;=\; L\, c^*_{\mathbb{R}^N, \frac{1}{L^2}A, \frac{1}{L}q, f}(e). \tag{2.8.67}$$

Taking $\beta = 1/L^2$, then

$$v_t(t,x,y) = \beta\, \nabla\cdot(A(y)\nabla v)(t,x,y) + \sqrt{\beta}\, q_1(y)\, \partial_x v(t,x,y) + f(x,y,v) \text{ over } \mathbb{R}\times\mathbb{R}^N.$$

Owing to Theorem 2.6.1, the function $\beta \mapsto c^*_{\mathbb{R}^N,\beta A,\sqrt{\beta}\,q,\,f}(e)/\sqrt{\beta}$ is decreasing in $\beta > 0$. Besides, $L \mapsto 1/L^2$ is decreasing in $L > 0$. Together with (2.8.67), one obtains that the function $L \mapsto c^*_{\mathbb{R}^N, A_L, q_L, f_L}(e)$ is increasing in $L > 0$ which completes the proof of Theorem 2.6.3. □

Proof of Theorem 2.6.5. Referring to Theorem 3.1.11, it follows that for each $B > 0$, we have:

$$\frac{c^*_{\Omega,A,\sqrt{B}\,q,Bf}(e)}{\sqrt{B}} = \min_{\lambda>0} \frac{k_{\Omega,e,\,A,\,\sqrt{B}\,q,B\zeta}(\lambda)}{\lambda\sqrt{B}}.$$

Owing to the same justifications explained in the proof of Theorem 2.6.1, $k_{\Omega,e,A,\sqrt{B}q,B\zeta}(\lambda)$

is the first eigenvalue of the problem

$$\begin{cases} \nabla\cdot(A(y)\nabla\psi(y)) + \Big[\lambda^2 e\cdot Ae - \lambda\sqrt{B}q_{_1}(y) + B\zeta(y)\Big]\psi(y) = k_{\Omega,e,A,\sqrt{B}q,B\zeta}(\lambda)\psi \text{ in } \omega, \\ \nu(x,y)\cdot A(y)\nabla\psi(y) = (0;\nu_\omega(y))\cdot A(y)\nabla\psi(y) = 0 \text{ on } \mathbb{R}\times\partial\omega. \end{cases} \tag{2.8.68}$$

For each $\lambda>0$ and $B>0$, let $\lambda^{'}=\lambda/\sqrt{B}$ and $k_{\Omega,e,\,A,\,\sqrt{B}\,q,B\zeta}(\lambda)=\mu(\lambda^{'},B)$. The first eigenvalue $\mu(\lambda^{'},B)$ has the following characterization:

$$\forall\lambda^{'}>0,\,\forall B>0,\ -\frac{\mu(\lambda^{'},B)}{\lambda^{'}B} = \min_{\substack{\varphi\in H^1(C)\setminus\{0\};\\ ||\varphi||_{L^2(C)}=1}} \frac{\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi\,dy}{\lambda^{'}B} + \int_C q_{_1}\varphi^2 - \lambda^{'}\int_C eAe\varphi^2 - \frac{\displaystyle\int_C\zeta(y)\varphi^2(y)\,dy}{\lambda^{'}} \tag{2.8.69}$$

$$= \min_{\substack{\varphi\in H^1(C)\setminus\{0\}\\ ||\varphi||_{L^2(C)}=1}} R(\lambda^{'},B,\varphi).$$

On the other hand, $B\mapsto R(\lambda^{'},B,\varphi)$ is decreasing in $B>0$. Consequently, fixing $\lambda^{'}>0$ and taking $0<B<B^{'}$,

$$-\frac{\mu(\lambda^{'},B)}{\lambda^{'}B} = R(\lambda^{'},B,\psi^{\lambda^{'},B}) > R(\lambda^{'},B^{'},\psi^{\lambda^{'},B}) \geq \min_{\substack{\varphi\in H^1(C)\setminus\{0\};\\ ||\varphi||_{L^2(C)}=1}} R(\lambda^{'},B^{'},\varphi) = -\frac{\mu(\lambda^{'},B^{'})}{\lambda^{'}B^{'}}.$$

In other words, for all $\lambda^{'}>0$, the function $B\mapsto\mu(\lambda^{'},B)/\lambda^{'}B$ is increasing in $B>0$. Now, we take randomly $0<B<B^{'}$. Thus,

$$\frac{c^*_{\Omega,A,\sqrt{B^{'}}\,q,B^{'}f}(e)}{\sqrt{B^{'}}} = \min_{\lambda^{'}>0}\frac{\mu(\lambda^{'},B^{'})}{\lambda^{'}B^{'}} = \frac{\mu(\lambda^{'}_{B^{'}},B^{'})}{\lambda^{'}_{B^{'}}\times B^{'}} > \frac{\mu(\lambda^{'}_{B^{'}},B)}{\lambda^{'}_{B^{'}}\times B} \geq \min_{\lambda^{'}>0}\frac{\mu(\lambda^{'},B)}{\lambda^{'}B} = \frac{c^*_{\Omega,A,\sqrt{B}\,q,Bf}(e)}{\sqrt{B}},$$

which means that $B\mapsto c^*_{\Omega,A,\sqrt{B}\,q,Bf}(e)/\sqrt{B}$ is increasing in $B>0$. □

2.9 Applications to homogenization problems

The reaction-advection-diffusion problem set in a heterogenous periodic domain Ω satisfying (3.1.2) generates a homogenization problem:

Let $e \in \mathbb{R}^d$ be a vector of unit norm. Assume that Ω, A, q, and f are $(L_1, \ldots, L_d)-$periodic and that they satisfy (3.1.2), (3.1.3), (3.1.4), (2.2.4) and (4.1.5).

For each $\varepsilon > 0$, let $\Omega^\varepsilon = \varepsilon\,\Omega$ and consider the following re-scales:

$$\forall (x,y) \in \Omega^\varepsilon, \quad A_\varepsilon(x,y) = A\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right),\, q_{\varepsilon}(x,y) = q\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), \text{ and } f_{\varepsilon}(x,y) = f\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

The coefficients A_ε, q_{ε}, and f_{ε} together with the domain Ω^ε are $(\varepsilon\, L_1, \ldots, \varepsilon\, L_d)-$periodic, and they satisfy similar properties to those of A, q, f and Ω.

Consider the parametric reaction-advection-diffusion problem

$$(P_\varepsilon)\begin{cases} u^\varepsilon_t(t,x,y) = \nabla\cdot(A_\varepsilon \nabla u^\varepsilon)(t,x,y) \ + \ q_\varepsilon \cdot \nabla u^\varepsilon \ + \ f_\varepsilon(x,y,u^\varepsilon),\ t\in\mathbb{R},\ (x,y)\in\Omega^\varepsilon, \\ \\ \nu^\varepsilon \cdot A_\varepsilon\ \nabla u^\varepsilon(t,x,y) = 0, \quad t\,\in\,\mathbb{R},\ (x,y)\,\in\,\partial\Omega^\varepsilon, \end{cases}$$

where $\nu^\varepsilon(x,y)$ denotes the outward unit normal on $\partial\Omega^\varepsilon$ at the point (x,y).

Owing to the results found by Berestycki and Hamel in section 6 of [2], and since the coefficients A_ε, f_{ε} and q_ε together with the domain Ω^ε satisfy all the necessary assumptions, it follows that the problem (P_ε) admits a minimal speed of propagation $c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,f_\varepsilon}(e) > 0$ such that (P_ε) has a solution u^ε in the form of a pulsating front within a speed c if and only if $c \geq c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,f_\varepsilon}(e) > 0$.

In this section, we investigate the limit of the parametric minimal speeds $c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,f_\varepsilon}(e)$ (whose parameter is ε) of the problems $(P_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0^+$. In other words, we search the limit of these minimal speeds as the periodicity cell $C^\varepsilon = \varepsilon\, C$ becomes a very small size. On the other hand, we study although not the most general setting, the variation of the map $\varepsilon \mapsto c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,f_\varepsilon}(e)$ in $\varepsilon > 0$.

Theorem 2.9.1 *Let $e \in \mathbb{R}^d$ be a unit vector, and let $\Omega \subseteq \mathbb{R}^N$ be a domain which is $L-$periodic and satisfying (3.1.2). Assume that $A = A(x,y)$, $q = q(x,y)$, and $f = f(x,y,u)$ are $L-$periodic and that they satisfy (3.1.3), (3.1.4), (2.2.4) and (4.1.5) together with the assumptions $\nabla . A\tilde{e} \equiv 0$ on $\overline{\Omega}$ and $\nu . A\tilde{e} = 0$ on $\partial\Omega$. For each $\varepsilon > 0$, consider the problem*

$$\begin{cases} u^\varepsilon_t(t,x,y) = \nabla\cdot(A_\varepsilon \nabla u^\varepsilon)(t,x,y) \ + \ q_\varepsilon \cdot \nabla u^\varepsilon \ + \ f_\varepsilon(x,y,u^\varepsilon),\ t\in\mathbb{R},\ (x,y)\in\Omega^\varepsilon \\ \nu^\varepsilon \cdot A_\varepsilon\ \nabla u^\varepsilon(t,x,y) = 0, \quad t\,\in\,\mathbb{R},\ (x,y)\,\in\,\partial\Omega^\varepsilon, \end{cases} \tag{2.9.1}$$

where A_ε, f_ε and q_ε are the coefficients defined in the beginning of this section. Then, the minimal speed $c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,\,f_\varepsilon}(e)$ of pulsating travelling fronts propagating in the direction of e and solving (2.9.1) satisfies

$$\lim_{\varepsilon\to 0^+} c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,\,f_\varepsilon}(e) = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\,\sqrt{\fint_C \zeta(x,y)dx\,dy}, \tag{2.9.2}$$

where C is the periodicity cell of Ω and $\tilde{e} = (e, 0, \cdots, 0) \in \mathbb{R}^N$.

Proof. As a first notice, we mention that the domain Ω^ε is the image of Ω by the a dilation whose center is the origin $O(0,\ldots,0)$ and whose scale factor is equal to ε. Consequently,

$$\text{for each } \varepsilon > 0,\ (\varepsilon x,\, \varepsilon y) \in \Omega^\varepsilon \quad \text{if and only if} \quad (x,y) \in \Omega, \text{ and}$$

$$(\varepsilon x,\, \varepsilon y) \in \partial\Omega^\varepsilon \quad \text{if and only if} \quad (x,y) \in \partial\Omega.$$

Moreover,

$$\forall \varepsilon > 0,\ \forall (x,y) \in \partial\Omega,\ \nu^\varepsilon(\varepsilon x,\, \varepsilon y) = \nu(x,y).$$

Consider now, for each $\varepsilon > 0$, the following change of variables

$$v^\varepsilon(t,x,y) = u^\varepsilon(t,\varepsilon x,\varepsilon y)\,; \quad (t,x,y) \in \mathbb{R}\times\Omega.$$

One gets

$$\forall (t,x,y) \in \mathbb{R}\times\Omega,\ v^\varepsilon_t(t,x,y) = u^\varepsilon_t(t,\varepsilon x,\varepsilon y),$$

$$\nabla_{x,y}\cdot(A(x,y)\nabla v^\varepsilon)(t,x,y) = \nabla_{x,y}\cdot(A_\varepsilon\nabla u^\varepsilon)(t,\varepsilon x,\varepsilon y) = \varepsilon^2\,\nabla\cdot(A_\varepsilon\nabla u^\varepsilon)(t,\varepsilon x,\varepsilon y),$$

and

$$\begin{aligned}\nu_\varepsilon(\varepsilon x,\varepsilon y)\cdot\left[A_\varepsilon\nabla u^\varepsilon\right](t,\varepsilon x,\varepsilon y) &= \nu(x,y)\cdot A\left(\frac{\varepsilon x}{\varepsilon},\frac{\varepsilon y}{\varepsilon}\right)\nabla u^\varepsilon(t,\varepsilon x,\varepsilon y)\\ &= \frac{1}{\varepsilon}\,\nu(x,y)\cdot A(x,y)\nabla v^\varepsilon(t,x,y) \text{ on } \mathbb{R}\times\partial\Omega.\end{aligned} \tag{2.9.3}$$

The boundary condition in (2.9.1) yields that $\nu_\varepsilon(\varepsilon x,\varepsilon y)\cdot\left[A_\varepsilon\nabla u^\varepsilon\right](t,\varepsilon x,\varepsilon y) = 0,$ for all $(t,x,y)\in\mathbb{R}\times\partial\Omega$ (which is equivalent to say: for all $(t,\varepsilon x,\varepsilon y)\in\mathbb{R}\times\partial\Omega^\varepsilon$). It follows from (2.9.3) that

$$\forall (t,x,y) \in \mathbb{R}\times\partial\Omega, \quad \nu\cdot A\nabla v^\varepsilon(t,x,y) = 0.$$

One can now conclude that: for each $\varepsilon > 0,$ u^ε satisfies (2.9.1) if and only if v^ε

satisfies

$$\begin{cases} v_t^\varepsilon(t,x,y) = \dfrac{1}{\varepsilon^2}\nabla\cdot(A\nabla v^\varepsilon)(t,x,y) \; + \; \dfrac{1}{\varepsilon}\, q\cdot\nabla v^\varepsilon \; + \; f(x,y,v^\varepsilon),\; t\in\mathbb{R},\; (x,y)\in\Omega, \\ \nu\cdot A\,\nabla v^\varepsilon(t,x,y) = 0,\; t\,\in\,\mathbb{R},\; (x,y)\,\in\,\partial\Omega. \end{cases} \tag{2.9.4}$$

Having the assumptions (3.1.2), (3.1.3), (3.1.4), (2.2.4), and (4.1.5) on $\Omega,\ A,\ q,$ and f, one gets that problem (2.9.4) admits, for each $\varepsilon>0$, a minimal speed of propagation denoted by $c^*_{\Omega,\,\left(\frac{1}{\varepsilon}\right)^2 A,\,\frac{1}{\varepsilon}\,q,\,f}(e)$.

Moreover, due to the change of variables between u^ε and v^ε, it follows that for each $\varepsilon>0,\ u^\varepsilon$ is a pulsating travelling front propagating in the direction of e within a speed c and solving (2.9.1) if and only if v^ε is a pulsating travelling front propagating in the direction of e within a speed $\dfrac{c}{\varepsilon}$ and solving (2.9.4). This yields that

$$\forall\varepsilon>0,\quad c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,\,f_\varepsilon}(e) \;=\varepsilon\, c^*_{\Omega,\,\left(\frac{1}{\varepsilon}\right)^2 A,\,\frac{1}{\varepsilon}\,q,\,f}(e) = c^*_{\Omega,\,MA,\,\sqrt{M}\,q,\,f}(e)/\sqrt{M}, \tag{2.9.5}$$

where $M=(\,1/\varepsilon\,)^2$.

As $\varepsilon\to0^+$, the variable $M\to+\infty$. Applying Theorem 2.4.1, with $\gamma=\dfrac{1}{2}$, one gets that

$$\lim_{M\to+\infty}\frac{c^*_{\Omega,\,MA,\,\sqrt{M}\,q,\,f}(e)}{\sqrt{M}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\;\sqrt{\fint_C \zeta(x,y)dx\,dy}.$$

Therefore, $\displaystyle\lim_{\varepsilon\to0^+} c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,\,f_\varepsilon}(e) = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\;\sqrt{\fint_C \zeta(x,y)dx\,dy}$, and the proof of Theorem 2.9.1 is complete. $\square$

Remark 2.9.2 *It is worth noticing that, in formula 2.9.2, the homogenized speed depends on the averages of the diffusion and reaction coefficients, but it does not depend on the advection.*

We move now to study the variation of the map $\varepsilon\mapsto c^*_{\Omega^\varepsilon,\,A_\varepsilon,\,q_\varepsilon,\,f_\varepsilon}(e)$ with respect to $\varepsilon>0$. In other words, we want to check the monotonicity behavior of the parametric minimal speed of propagation, whose parameter $\varepsilon>0$, as the periodicity cell of the domain of propagation shrinks or enlarges within a ratio ε. In this study, we will consider the same situation of Theorem 2.6.1 and also the same notations introduced in the beginning of section 2.9:

Theorem 2.9.3 *Let $e=(1,0\ldots,0)$. Assume that Ω has the form $\mathbb{R}\times\omega$ where ω may or may not be bounded (precisely described in section 2.3) and that the diffusion matrix*

$A = A(y)$ satisfies (2.3.5) together with the assumption that e is an eigenvector of $A(y)$ for all $y \in \overline{\omega}$, that is

$$A(x,y)e = A(y)e = \alpha(y)e, \text{ for all } (x,y) \in \mathbb{R} \times \overline{\omega}; \tag{2.9.6}$$

where $y \mapsto \alpha(y)$ is a positive $(L_1, \ldots, L_d)-$ periodic function defined over $\overline{\omega}$. The nonlinearity f is assumed to satisfy (2.3.3) and (2.3.4). Assume further more that the advection field q (when it exists) is in the form $q(x,y) = (q_1(y), 0, \ldots, 0)$ where q_1 has a zero average over C, the periodicity cell of ω. For $\varepsilon > 0$ consider the reaction-advection-diffusion problem

$$\begin{cases} \forall\, t \in \mathbb{R},\ \forall\,(x,y) \in \Omega^\varepsilon = \mathbb{R} \times \varepsilon\,\omega, \\ u_t^\varepsilon(t,x,y) = \nabla \cdot (A_\varepsilon \nabla u^\varepsilon)(t,x,y)\ +\ q_\varepsilon \cdot\ \nabla u^\varepsilon\ +\ f_\varepsilon(x,y,u^\varepsilon); \\ \nu^\varepsilon \cdot A_\varepsilon\ \nabla u^\varepsilon(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega^\varepsilon. \end{cases} \tag{2.9.7}$$

Then, the map $\varepsilon \mapsto c^*_{\Omega^\varepsilon,\, A_\varepsilon,\, q_\varepsilon,\, f_\varepsilon}(e)$ is increasing in $\varepsilon > 0$.

Proof of Theorem 2.9.3. For each $\varepsilon > 0$, we consider the change of variables

$$v^\varepsilon(t,x,y) = u^\varepsilon(t, \varepsilon x, \varepsilon y)\,; \quad (t,x,y) \in \mathbb{R} \times \Omega.$$

Owing to the justifications shown in the proof of Theorem 2.9.1, one consequently obtains

$$\forall \varepsilon > 0, \quad c^*_{\Omega^\varepsilon,\, A_\varepsilon,\, q_\varepsilon,\, f_\varepsilon}(e) \ = \varepsilon\, c^*_{\Omega,\, \left(\frac{1}{\varepsilon}\right)^2 A,\, \frac{1}{\varepsilon}\, q,\, f}(e) = c^*_{\Omega,\, \beta A,\, \sqrt{\beta}\, q,\, f}(e)/\sqrt{\beta}, \tag{2.9.8}$$

where $\beta(\varepsilon) = (\,1/\varepsilon\,)^2$.

Applying Theorem 2.6.1, it follows that the map $\eta_1 :\ \beta \mapsto c^*_{\Omega,\, \beta A,\, \sqrt{\beta}\, q,\, f}(e)/\sqrt{\beta}$ is decreasing in $\beta > 0$. On the other hand, the map $\eta_2 :\ \varepsilon \mapsto \beta(\varepsilon)$ is also decreasing in $\varepsilon > 0$. Therefore, $\varepsilon \mapsto c^*_{\Omega^\varepsilon,\, A_\varepsilon,\, q_\varepsilon,\, f_\varepsilon}(e)$, which is the composition $\eta_1 \circ \eta_2$, is increasing in $\varepsilon > 0$ and this completes our proof. □

Other homogenization results, concerning reaction-advection-diffusion problems, were given in the case of a combustion-type nonlinearity $f = f(u)$ satisfying

$$\begin{cases} \exists\, \theta\ \in\ (0,1),\ f(s) = 0 \text{ for all } s \in [0,\theta]\,,\ f(s) > 0 \text{ for all } s \in (\theta,1),\ f(1) = 0, \\ \exists \rho \in (0, 1-\theta), \quad f \text{ is non-increasing on } \ [1-\rho, 1]\,. \end{cases} \tag{2.9.9}$$

Consider the equation

$$u_t^\varepsilon(t,x) = \nabla\cdot(A(\varepsilon^{-1}x)\nabla u^\varepsilon) \;+\; \varepsilon^{-1}q(\varepsilon^{-1}x)\cdot\nabla u^\varepsilon \;+\; f(u^\varepsilon) \quad \text{in } \mathbb{R}^N, \tag{2.9.10}$$

where the nonlinearity f satisfies (2.9.9), and the drift and diffusion coefficients q and A satisfy the general assumptions (3.1.3) and (3.1.4), with periodicity 1 in all variables $x_1,\ldots,x_N$. Fix a unit vector e of $\mathbb{R}^N$. From Berestycki and Hamel [2], it follows that for each $\varepsilon>0$, problem (2.9.10) admits a unique pulsating front $(c_\varepsilon, u^\varepsilon)$ such that

$$u^\varepsilon(t,x) = \phi^\varepsilon(x\cdot e + c_\varepsilon t,\; x)$$

where $\phi^\varepsilon(s,x)$ is $(\varepsilon,\ldots,\varepsilon)-$periodic in x that satisfies $\phi^\varepsilon(-\infty,.)=0$ and $\phi^\varepsilon(+\infty,.)=1$. The functions u^ε are actually unique up to shifts in time, and one can assume that $\max_{\mathbb{R}^N}\phi^\varepsilon(0,.)=\theta$.

Concerning problem (2.9.10), Heinze [15] proved that

$$\text{as } \;\varepsilon\to 0^+,\; c_\varepsilon\to c_0>0, \text{ and } u^\varepsilon(t,x)\to u_0(x\cdot e + c_0 t) \text{ weakly in } H^1_{loc},$$

where (c_0,u_0) is the unique solution of the one-dimensional homogenized equation

$$\begin{cases} a^*\,u_0^{''} - c_0 u_0^{'} + f(u_0) = 0 \text{ in } \mathbb{R}, \\ u_0(-\infty)=0<u_0<u_0(+\infty)=1 \text{ in } \mathbb{R},\; u_0(0)=\theta \end{cases} \tag{2.9.11}$$

and a^* is a positive constant determined in [15].

In Theorem 1 of Caffarelli, Lee, Mellet [10], the homogenization limit was combined with the singular high activation limit for the reaction (one can also see [11] in this context) while the diffusion matrix was taken $A = Id_{\mathbb{R}^N}$. More precisely, the nonlinearity had the form $f_\varepsilon(u) = \dfrac{1}{\varepsilon}\beta(\dfrac{u}{\varepsilon})$ with $\beta(s)$ a Lipschitz fucntion satisfying

$$\beta(s)>0 \text{ in } (0,1) \text{ and } \beta(s)=0 \text{ otherwise.}$$

These nonlinearities approach a Dirac mass at $u=1$.

2.10 Open problems

In all the results of this paper, we deal with nonlinearities of the "KPP" type. In the periodic framework of this paper, pulsating travelling fronts exist also with other types of nonlinearities (see Theorems 1.13 and 1.14 in [2]). Namely, they exist when

$f = f(x, y, u)$ is of the "combustion" type satisfying:

$$\begin{cases} f \text{ is globally Lipschitz-continuous in } \overline{\Omega} \times \mathbb{R}, \\ \forall\, (x, y) \in \overline{\Omega}, \forall\, s \in (-\infty, 0] \cup [1, +\infty),\ f(s, x, y) = 0, \\ \exists\, \rho \in\ (0, 1),\ \forall (x, y)\ \in \overline{\Omega},\ \forall\, 1 - \rho \leq\ s\ \leq s'\ \leq\ 1,\ f(x, y, s)\ \geq\ f(x, y, s'), \end{cases} \tag{2.10.1}$$

and

$$\begin{cases} f \ \text{ is } L\text{−periodic with respect to } x, \\ \exists\, \theta\ \in\ (0, 1),\ \forall (x, y) \in \overline{\Omega},\ \forall\, s \in [0, \theta],\ f(x, y, s) = 0, \\ \forall\, s \in\ (\theta, 1),\ \exists\, (x, y) \in \overline{\Omega} \text{ such that } f(x, y, s)\ >\ 0, \end{cases} \tag{2.10.2}$$

or when $f = f(x, y, u)$ is of the "ZFK" (for Zeldovich-Frank- Kamenetskii) type satisfying (2.10.1) and

$$\begin{cases} f \ \text{ is } L\text{−periodic with respect to } x, \\ \exists \delta > 0, \text{ the restriction of } f \text{ to } \overline{\Omega}\ \times\ [0, 1] \text{ is of class } C^{1,\,\delta}, \\ \forall\, s \in\ (0, 1),\ \exists\, (x, y) \in \overline{\Omega} \text{ such that } f(x, y, s)\ >\ 0. \end{cases} \tag{2.10.3}$$

In particular, the "KPP" nonlinearities are of the "ZFK" type.

Recently, El Smaily [12] gave $\min - \max$ and $\max - \min$ formulæ for the speeds of propagation of problem (4.1.8) taken with a "ZFK" or a "combustion" nonlinearity. These formulæ, together with the results of this paper, can give important estimates for the parametric minimal speeds of the problem (4.1.8) when f is a "ZFK" nonlinearity which is not of the "KPP" type. Indeed, if f is a "ZFK" nonlinearity, one can find a "KPP" function $h = h(x, y, u)$ such that

$$\forall (x, y, u) \in \overline{\Omega} \times \mathbb{R},\ f(x, y, u) \leq h(x, y, u).$$

Referring to formula (1.17) in El Smaily [12], one can conclude that

$$\forall M > 0, \forall B > 0,\ \forall \gamma \in \mathbb{R},\quad c^*_{\Omega, MA,\, M^\gamma\, q,\, Bf}(e) \leq c^*_{\Omega, MA,\, M^\gamma\, q,\, Bh}(e).$$

Moreover, if f is a "ZFK" nonlinearity satisfying the additional assumption

$$\forall (x, y) \in \overline{\Omega},\ f'_u(x, y, 0) > 0, \tag{2.10.4}$$

then one can find a "KPP" function $g = g(x, y, u)$ such that $g \leq f$ in $\overline{\Omega} \times \mathbb{R}$, and thus

$$\begin{array}{c}\forall M > 0, \forall B > 0, \ \forall \gamma \in \mathbb{R}, \\ c^*_{\Omega, MA,\, M^\gamma q,\, Bg}(e) \leq c^*_{\Omega, MA,\, M^\gamma q,\, Bf}(e) \leq c^*_{\Omega, MA,\, M^\gamma q,\, Bh}(e).\end{array} \tag{2.10.5}$$

As a consequence, under the assumptions that $0 \leq \gamma \leq 1/2$, $\nu \cdot A\tilde{e} = 0$ on $\partial\Omega$, and $\nabla \cdot A\tilde{e} \equiv 0$ in Ω, Theorem 2.4.1 implies that

$$\limsup_{M \to +\infty} \frac{c^*_{\Omega,\, MA,\, M^\gamma q,\, f}(e)}{\sqrt{M}} \leq 2 \sqrt{\fint_C \tilde{e} A \tilde{e}(x, y) dx\, dy} \sqrt{\fint_C g'_u(x, y, 0) dx\, dy}, \tag{2.10.6}$$

and

$$\liminf_{M \to +\infty} \frac{c^*_{\Omega,\, MA,\, M^\gamma q,\, f}(e)}{\sqrt{M}} = 2 \sqrt{\fint_C \tilde{e} A \tilde{e}(x, y) dx\, dy} \sqrt{\fint_C h'_u(x, y, 0) dx\, dy} > 0. \tag{2.10.7}$$

If f is a "combustion" nonlinearity, then problem (4.1.8) admits a solution (c, u) where $c = c_{\Omega,A,q,f}(e) > 0$ is unique and $u = u(t, x, y)$ is increasing in t and it is unique up to a translation in t. Taking g as a "KPP" nonlinearity such that $g \geq f$ in $\overline{\Omega} \times \mathbb{R}$ and using Theorem 2.4.1, it follows that

$$\begin{array}{c}\displaystyle\limsup_{M \to +\infty} \frac{c_{\Omega,\, MA,\, M^\gamma q,\, f}(e)}{\sqrt{M}} \leq 2 \sqrt{\fint_C \tilde{e} A \tilde{e}(x, y) dx\, dy} \sqrt{\fint_C g'_u(x, y, 0) dx\, dy} \\ \displaystyle\text{together with} \quad \liminf_{M \to +\infty} \frac{c_{\Omega,\, MA,\, M^\gamma q,\, f}(e)}{\sqrt{M}} \geq 0.\end{array} \tag{2.10.8}$$

Similarly, one can get several estimates concerning the case of a small diffusion factors, small (resp. large) reaction factors, or small (resp. large) periodicity parameters.

The above motivation gives several upper and lower estimates for the parametric speeds of propagation. However, the exact limits are not known. This leads us to ask about the asymptotics of the minimal speeds of propagation with respect to diffusion, reaction and periodicity factors in the "ZFK" case and about the asymptotics of the unique parametric speed of propagation in the "combustion" case. These studies should help, as it was done in section 2.9, in solving some homogenization problems in the "ZFK" case.

Besides, Theorem 2.9.1 gives the limit of $c^*_{\Omega^\varepsilon,\, A_\varepsilon,\, q_\varepsilon,\, f_\varepsilon}(e)$ as $\varepsilon \to 0^+$. However, finding the homogenized equation of (2.9.1) in the "KPP" remains an open problem.

2.11 Conclusions

As we mentioned in the beginning of this paper, our first aim was to give a complete and rigorous analysis of the *minimal speed of propagation* of pulsating travelling fronts solving *parametric heterogeneous* reaction-advection-diffusion equations in a periodic framework. In the paper of Berestycki, Hamel and Nadirashvili [3], several upper and lower estimates for the parametric minimal speed of propagation were given (see Theorems 1.6 and 1.10 in [3]). However, the exact asymptotic behaviors of the minimal speed with respect to diffusion and reaction factors and with respect to the periodicity parameter L were not given there. In this paper, we determined the exact asymptotes of the minimal speed in the "KPP" periodic framework. In sections 2.3, 2.4 and 2.5, we proved that (under some assumptions on A, q, f and Ω) the asymptotes of the parametric minimal speed are either

$$2\sqrt{\max_{\overline{\omega}}\zeta}\sqrt{\max_{\overline{\omega}} eAe} \quad \text{or} \quad 2\sqrt{\displaystyle{\int\!\!\!\!\!\!-}_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\sqrt{\displaystyle{\int\!\!\!\!\!\!-}_C \zeta(x,y)dx\,dy}.$$

(see Theorems 2.3.1, 2.3.4, 2.4.1, 2.4.3, 2.5.1 and 2.5.2 above). Moreover, we found in section 2.3 that the presence of an advection field, in the general form or in the form of shear flows, changes the asymptotic behavior of the minimal speed within a small diffusion (see Theorem 2.3.3 and Remark 2.3.6). Conversely, we proved in section 4 that the presence of a general advection field $M^\gamma q$ (where q satisfies (3.1.4)) has no effect on $\displaystyle\lim_{M\to+\infty}\frac{c^*_{\Omega,MA,\,M^\gamma q,\,f}(e)}{\sqrt{M}}$ whenever $0\leq\gamma\leq 1/2$ (see Theorem 2.4.1). Furthermore, we studied, in a particular periodic framework, the variations of the maps $\beta\mapsto\dfrac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}}$ and $L\mapsto c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$ and $B\mapsto\dfrac{c^*_{\Omega,A,\sqrt{B}\,q,Bf}(e)}{\sqrt{B}}$ with respect to the positive variables β, L and B respectively. Roughly speaking, we found that the first and the third maps have opposite senses of variations (see Theorems 2.6.1 and 2.6.5). On the other hand, Theorem 2.6.3 and Theorem 2.9.3 yield that the minimal speed increases when the medium undergoes a dilation whose scale factor is greater than 1.

The second aim was to find the homogenized "KPP" minimal speed. We achieved this goal in section 2.9 (Theorem 2.9.1) under the assumptions of free divergence on $A(x,y)\tilde{e}$ and invariance of the domain in the direction $A(x,y)\tilde{e}$. This was an application to the results obtained in section 2.4. The found homogenized speed should play an important role in finding the homogenized reaction-advection-diffusion equation in the "KPP" case. In a forthcoming paper [13], we find also the homogenized speed in the one dimensional case but in a more general setting (in fact, the assumption of divergence free is equivalent to the assumption that the diffusion term $x \mapsto a(x)$ is constant over $\mathbb{R}$ in the case $N = 1$).

All the mathematical results obtained in this paper can be applied to study some *spreading* phenomena. Referring to the results of Weinberger [30], one can conclude that the *spreading speed* is equal to the "KPP" *minimal speed of propagation* in the periodic framework under some assumptions on the initial data $u_0 := u_0(x, y) = u(0, x, y)$ which is defined on a periodic domain Ω of $\mathbb{R}^N$. In such a setting, all our results can be applied to give rigorous answers on the asymptotic behavior of the *parametric spreading speed* with respect to diffusion and reaction factors and with respect to the periodicity parameter.

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CHAPTER 3

Min-Max formulæ for the speeds of pulsating travelling fronts in periodic excitable media

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Abstract. This paper is concerned with some nonlinear propagation phenomena for reaction-advection-diffusion equations in a periodic framework. It deals with travelling wave solutions of the equation

$$u_t = \nabla \cdot (A(z)\nabla u) \; + q(z) \cdot \nabla u + \; f(z,u), \quad t \,\in \mathbb{R}, \; z \,\in \,\Omega,$$

propagating with a speed c. In the case of a "combustion" nonlinearity, the speed c exists and it is unique, while the front u is unique up to a translation in t. We give a $\min - \max$ and a $\max - \min$ formula for this speed c. On the other hand, in the case of a "ZFK" or a "KPP" nonlinearity, there exists a minimal speed of propagation c^*. In this situation, we give a $\min - \max$ formula for c^*.

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3.1 Introduction and main results

3.1.1 A description of the periodic framework

The goal of this paper is to give some variational formulæ for the speeds of pulsating travelling fronts corresponding to reaction-diffusion-advection equations set in a heterogenous periodic framework. In fact, many works, such as Hamel [7], Heinze, Papanicolaou, Stevens [10], and Volpert, Volpert, Volpert [17] treated this problem in simplified situations and under more strict assumptions. In this paper, we treat the problem in the most general periodic framework. We are concerned with equations of the type

$$\begin{cases} u_t = \nabla\cdot(A(z)\nabla u) \ + q(z)\cdot\nabla u + \ f(z,u), \quad t\in\mathbb{R},\ z\in\Omega, \\ \nu\cdot A\ \nabla u(t,z) = 0, \quad t\in\mathbb{R},\ z\in\partial\Omega, \end{cases} \tag{3.1.1}$$

where $\nu(z)$ is the unit outward normal on $\partial\Omega$ at the point z. In this context, let us detail the mathematical description of the heterogeneous setting.

Concerning the domain, let $N\geq 1$ be the space dimension, and let d be an integer so that $1\leq\ d\leq\ N$. For an element $z=(x_1,x_2,\cdots,x_d,x_{d+1},\cdots,x_N)\in\mathbb{R}^N$, we denote by $x\ =\ (x_1,x_2,\cdots,x_d)$ and by $y=(x_{d+1},\cdots,x_N)$ the two tuples so that $z=(x,y)$. Let $L_1,\cdots,L_d$ be d positive real numbers, and let Ω be a C^3 non empty connected open subset of $\mathbb{R}^N$ satisfying

$$\begin{cases} \exists\, R\geq 0\,;\forall\,(x,y)\ \in\ \Omega,\ |y|\ \leq\ R, \\ \forall\,(k_1,\cdots,k_d)\ \in\ L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}, \quad \Omega\ =\ \Omega+\displaystyle\sum_{k=1}^{d}k_ie_i, \end{cases} \tag{3.1.2}$$

where $(e_i)_{1\leq i\leq N}$ is the canonical basis of $\mathbb{R}^N$.

As $d\ \geq\ 1$, one notes that the set Ω satisfying (3.1.2) is unbounded. We have many archetypes of such a domain. The case of the whole space $\mathbb{R}^N$ corresponds to $d=N$, where $L_1,\ldots,L_N$ are any positive numbers. The case of the whole space $\mathbb{R}^N$ with a periodic array of holes can also be considered. The case $d=1$ corresponds to domains which have only one unbounded dimension, namely infinite cylinders which may be straight or have oscillating periodic boundaries, and which may or may not have periodic perforations. The case $2\leq d\leq N-1$ corresponds to infinite slabs.

In this periodic situation, we give the following definitions:

Definition 3.1.1 (Periodicity cell) *The set*

$$C \;=\; \{\,(x,y)\in\Omega;\; x_1\in(0,L_1),\cdots,x_d\in(0,L_d)\}$$

is called the periodicity cell of Ω.

Definition 3.1.2 (L-periodic fields) *A field* $w:\Omega\to\mathbb{R}^N$ *is said to be L-periodic with respect to x if*

$$w(x_1+k_1,\cdots,x_d+k_d\,,y) \;=\; w(x_1,\cdots,x_d,y)$$

almost everywhere in Ω, *and for all* $k=(k_1,\cdots,k_d)\in\displaystyle\prod_{i=1}^{d}L_i\mathbb{Z}$.

Let us now detail the assumptions concerning the coefficients in (3.1.1). First, the diffusion matrix $A(x,y)\;=\;(A_{ij}(x,y))_{1\le i,j\le N}$ is a symmetric $C^{2,\delta}(\,\overline{\Omega}\,)$ (with $\delta\;>\;0$) matrix field satisfying

$$\begin{cases} A \text{ is } L\text{-periodic with respect to } x,\\ \exists\, 0<\alpha_1\le\alpha_2;\forall(x,y)\;\in\;\Omega,\forall\,\xi\;\in\;\mathbb{R}^N,\\ \alpha_1|\xi|^2\;\le\;\displaystyle\sum_{1\le i,j\le N}A_{ij}(x,y)\xi_i\xi_j\;\;\le\alpha_2|\xi|^2. \end{cases} \tag{3.1.3}$$

The underlying advection $q(x,y)=(q_1(x,y),\cdots,q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta>0$) vector field satisfying:

$$\begin{cases} q \text{ is } L\text{-periodic with respect to }\; x,\\ \nabla\cdot q\;=0\quad\text{in }\overline{\Omega}\,,\\ q\cdot\nu\;=0\quad\text{on }\partial\Omega\,,\\ \forall\,1\le\;i\;\le\;d,\qquad\displaystyle\int_C q_i\,dx\,dy\;=\;0. \end{cases} \tag{3.1.4}$$

Lastly, let $f=f(x,y,u)$ be a function defined in $\overline{\Omega}\times\mathbb{R}$ such that

$$\begin{cases} f \text{ is globally Lipschitz-continuous in }\;\overline{\Omega}\times\mathbb{R},\\ \forall\,(x,y)\in\;\overline{\Omega},\;\forall\,s\in(-\infty,0]\cup[1,+\infty),\;f(s,x,y)=0,\\ \exists\,\rho\in\;(0,1),\;\forall(x,y)\;\in\overline{\Omega},\;\forall\,1-\rho\le\;s\;\le s'\;\le\;1,\;f(x,y,s)\;\ge\;f(x,y,s'). \end{cases} \tag{3.1.5}$$

One assumes that

$$f \text{ is } L\text{-periodic with respect to } x. \tag{3.1.6}$$

Moreover, the function f is assumed to be of one of the following two types: either

$$\begin{cases} \exists\, \theta \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall\, s \in [0,\theta],\ f(x,y,s) = 0, \\ \forall\, s \in (\theta,1),\ \exists\,(x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0, \end{cases} \tag{3.1.7}$$

or

$$\begin{cases} \exists\, \delta > 0, \text{ the restriction of } f \text{ to } \overline{\Omega} \times [0,1] \text{ is of class } C^{1,\delta}, \\ \forall\, s \in (0,1),\ \exists\,(x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0. \end{cases} \tag{3.1.8}$$

Definitions 3.1.3 *A nonlinearity f satisfying (3.1.5), (3.1.6) and (3.1.7) is called a "combustion" nonlinearity. The value θ is called the ignition temperature.*
A nonlinearity f satisfying (3.1.5), (3.1.6), and (3.1.8) is called a "ZFK" (for Zeldovich-Frank- Kamenetskii) nonlinearity.
If f is a "ZFK" nonlinearity that satisfies

$$f'_u(x,y,0) = \lim_{u \to 0^+} f(x,y,u)/u > 0, \tag{3.1.9}$$

with the additional assumption

$$\forall\,(x,y,s) \in \overline{\Omega} \times (0,1), \quad 0 < f(x,y,s) \leq f'_u(x,y,0) \times s, \tag{3.1.10}$$

then f is called a "KPP"(for Kolmogorov, Petrovsky, and Piskunov, see [12]) nonlinearity.

The simplest examples of "combustion" and "ZFK" nonlinearities are when $f(x,y,u) = f(u)$ where: either

$$\begin{cases} f \text{ is Lipschitz-continuous in } \mathbb{R}, \\ \exists\,\theta \in (0,1),\ f(s) = 0 \text{ for all } s \in (-\infty,\theta] \cup [1,+\infty), \\ \text{and } f(s) > 0 \text{ for all } s \in (\theta,1), \\ \exists\,\rho \in (0,1-\theta), \quad f \text{ is non-increasing on } [1-\rho,1], \end{cases} \tag{3.1.11}$$

or

$$\begin{cases} f \text{ is defined on } \mathbb{R},\ f \equiv 0 \text{ in } \mathbb{R} \setminus (0,1), \\ \exists\, \delta > 0, \text{ the restriction of } f \text{ on the interval } [0,1] \text{ is } C^{1,\delta}([0,1]), \\ f(0) = f(1) = 0, \text{ and } f(s) > 0 \text{ for all } s \in (0,1), \\ \exists\, \rho > 0, \quad f \text{ is non-increasing on } [1-\rho, 1]. \end{cases} \tag{3.1.12}$$

If $f = f(u)$ satisfies (3.1.11), then it is a homogeneous "combustion" nonlinearity. On the other hand, a nonlinearity $f = f(u)$ that satisfies (3.1.12) is homogeneous of the "ZFK" type. Moreover, a KPP homogeneous nonlinearity is a function $f = f(u)$ that satisfies (3.1.12) with the additional assumption

$$\forall\, s \in (0,1),\ 0 < f(s) \leq f'(0)\, s. \tag{3.1.13}$$

As typical examples of nonlinear heterogeneous sources satisfying (3.1.5-3.1.6) and either (3.1.7) or (3.1.8), one can consider the functions of the type

$$f(x,y,u) = h(x,y)\, g(u),$$

where h is a globally Lipschitz-continuous, positive, bounded, and $L-$periodic with respect to x function defined in $\overline{\Omega}$, and g is a function satisfying either (3.1.11) or (3.1.12).

Definition 3.1.4 (Pulsating fronts and speed of propagation) *Let $e = (e^1, \cdots, e^d)$ be an arbitrarily given unit vector in $\mathbb{R}^d$. A function $u = u(t,x,y)$ is called a pulsating travelling front propagating in the direction of $-e$ with an effective speed $c \neq 0$, if u is a classical solution of:*

$$\begin{cases} u_t = \nabla \cdot (A(x,y)\nabla u) + q(x,y) \cdot \nabla u + f(x,y,u),\ t \in \mathbb{R},\ (x,y) \in \Omega, \\ \nu \cdot A\, \nabla u(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega, \\ \forall\, k \in \prod_{i=1}^{d} L_i\mathbb{Z},\ \forall\,(t,x,y) \in \mathbb{R} \times \overline{\Omega},\ u(t + \dfrac{k \cdot e}{c}, x, y) = u(t, x+k, y), \\ \lim\limits_{x \cdot e \to -\infty} u(t,x,y) = 0, \text{ and } \lim\limits_{x \cdot e \to +\infty} u(t,x,y) = 1, \\ 0 \leq u \leq 1, \end{cases} \tag{3.1.14}$$

where the above limits hold locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e .

Several works were concerned with pulsating travelling fronts in periodic media (see

[1], [2], [11], [13], [15], [16], and [19]).

In the general periodic framework, we recall two essential known results and then we move to our main results.

Theorem 3.1.5 (Berestycki, Hamel [1]) *Let Ω be a domain satisfying (3.1.2), let e be any unit vector of $\mathbb{R}^d$ and let f be a nonlinearity satisfying (3.1.5-3.1.6) and (3.1.7). Assume, furthermore, that A and q satisfy (3.1.3) and (3.1.4) respectively. Then, there exists a classical solution (c, u) of (4.1.8). Moreover, the speed c is positive and unique while the function $u = u(t, x, y)$ is increasing in t and it is unique up to a translation. Precisely, if (c^1, u^1) and (c^2, u^2) are two classical solutions of (4.1.8), then $c^1 = c^2$ and there exists $h \in \mathbb{R}$ such that $u^1(t, x, y) = u^2(t + h, x, y)$ for all $(t, x, y) \in \mathbb{R} \times \overline{\Omega}$.*

In a periodic framework, Theorem 3.1.5 yields the existence of a pulsating travelling front in the case of a "combustion" nonlinearity with an ignition temperature θ. It implies, also, the uniqueness of the speed and of the profile of u. For "ZFK" nonlinearities, we have

Theorem 3.1.6 (Berestycki, Hamel [1]) *Let Ω be a domain satisfying (3.1.2), let e be any unit vector in $\mathbb{R}^d$ and let f be a nonlinearity satisfying (3.1.5-3.1.6) and (3.1.8). Assume, furthermore, that A and q satisfy (3.1.3) and (3.1.4) respectively. Then, there exists $c^*_{\Omega,A,q,f}(e) > 0$ such that the problem (4.1.8) has no solution (c, u) such that $u_t > 0$ in $\mathbb{R} \times \overline{\Omega}$ if $c < c^*_{\Omega,A,q,f}(e)$ while, for each $c \geq c^*_{\Omega,A,q,f}(e)$, it has a solution (c, u) such that u is increasing in t.*

In fact, the existence and the monotonicity of a solution $u^* = u^*(t, x, y)$ of (4.1.8) for $c = c^*_{\Omega,A,q,f}(e) > 0$ holds by approaching the "ZFK" nonlinearity f by a sequence of combustion nonlinearities $(f_\theta)_\theta$ such that $f_\theta \to f$ uniformly in $\mathbb{R} \times \overline{\Omega}$ as $\theta \searrow 0^+$ (see more details in step 2 of the proof of formula (3.1.17) below, section 3.4). It follows, from Theorem 3.1.5, that for each $\theta > 0$, there exists a solution (c_θ, u_θ) of (4.1.8) with the nonlinearity f_θ such that u_θ is increasing with respect to t. From parabolic estimates, the functions u_θ, converge up to a subsequence, to a function u^* in $C^2_{loc}(\mathbb{R} \times \overline{\Omega})$ as $\theta \to 0^+$. Moreover, Lemmas 6.1 and 6.2 in [1] yield the existence of a constant $c^*(e) = c^*_{\Omega,A,q,f}(e) > 0$ such that $c_\theta \nearrow c^*(e)$ as $\theta \searrow 0$. Hence, the couple $(c^*(e), u^*)$ becomes a classical solution of (4.1.8) with the nonlinearity f and one gets that u^* is nondecreasing with respect to t as a limit of the increasing functions u_θ. Finally, one applies the strong parabolic maximum principle and Hopf lemma to get that w is positive in $\mathbb{R} \times \overline{\Omega}$. In other words, u^* is increasing in $t \in \mathbb{R}$. Actually, in the "ZFK" case, under the additional non-degeneracy assumption (3.1.9), it is known that any pulsating front with speed c is increasing in time and $c \geq c^*(e)$ (see [1]).

The value $c^*_{\Omega,A,q,f}(e)$ which appears in Theorem 3.1.6 is called the minimal speed of propagation of the pulsating travelling fronts propagating in the direction $-e$ (satisfying the reaction-advection-diffusion problem (4.1.8)).

We mention that the uniqueness of the pulsating travelling fronts, up to shifts in time, for each $c \geq c^*_{\Omega,A,q,f}(e)$, has been proved recently by Hamel and Roques [8] for "KPP" nonlinearities. On the other hand, a variational formula for the minimal speed of propagation $c^*_{\Omega,A,q,f}(e)$, in the case of a KPP nonlinearity, was proved in Berestycki, Hamel, Nadirashvili [2]. This formula involves eigenvalue problems and gives the value of the minimal speed in terms of the domain Ω and in terms of the coefficients A, q, and f appearing in problem (4.1.8). The asymptotic behaviors and the variations of the minimal speed of propagation, as a function of the diffusion, advection and reaction factors and as a function of the periodicity parameters, were widely studied in Berestycki, Hamel, Nadirashvili [3], El Smaily [5], El Smaily, Hamel, Roques [6], Heinze [9], Ryzhik, Zlatoš [14], and Zlatoš [21].

3.1.2 Main results

In the periodic framework, having (in Theorems 3.1.5 and 3.1.6) the existence results and some qualitative properties of the pulsating travelling fronts propagating in the direction of a fixed unit vector $-e \in \mathbb{R}^d$, we search a variational formula for the unique speed of propagation $c = c(e)$ whenever f is of the "combustion" type, and for the minimal speed $c^* = c^*_{\Omega,A,q,f}(e)$ whenever f is of the "ZFK" or the "KPP" type. We will answer the above investigations in the following theorem, but before this, we introduce the following

Notation 3.1.7 *For each function $\phi = \phi(s,x,y)$ in $C^{1,\delta}(\mathbb{R}\times\overline{\Omega})$ (for some $\delta \in [0,1)$), let*

$$F[\phi] := \nabla_{x,y}\cdot(A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \nabla_{x,y}\cdot(A\tilde{e}\phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi) \text{ in } \mathcal{D}'(\mathbb{R}\times\Omega),$$

where $\tilde{e} = (e,0,\cdots,0) \in \mathbb{R}^N$ and e denotes a unit vector of $\mathbb{R}^d$.

The first main result deals with the "combustion" case.

Theorem 3.1.8 *Let e a unit vector of $\mathbb{R}^d$. Assume that Ω is a domain satisfying (3.1.2) and f is a nonlinear source satisfying (3.1.5) and (3.1.6). Assume furthermore that A*

and q satisfy (3.1.3) and (3.1.4) respectively. Consider the set of functions

$$\begin{aligned} E \;=\; & \big\{\varphi=\varphi(s,x,y),\ \varphi \text{ is of class } C^{1,\mu}(\mathbb{R}\times\overline{\Omega}) \text{ for each } \mu\in[0,1), F[\varphi]\in C(\mathbb{R}\times\overline{\Omega}),\\ & \varphi \text{ is } L\text{-periodic with respect to } x,\ \varphi_s>0 \text{ in } \mathbb{R}\times\overline{\Omega},\ \varphi(-\infty,.,.)=0,\\ & \varphi(+\infty,.,.)=1 \text{ uniformly in } \overline{\Omega}, \text{ and } \nu\cdot A(\nabla_{x,y}\varphi+\tilde{e}\varphi_s)=0 \text{ on } \mathbb{R}\times\partial\Omega\big\}. \end{aligned}$$

For each $\varphi\in E$, we define the function $R\varphi\in C(\mathbb{R}\times\overline{\Omega})$ as, for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$,

$$R\,\varphi(s,x,y)=\frac{F[\varphi](s,x,y)+\ q\cdot\nabla_{x,y}\varphi(s,x,y)+f(x,y,\varphi)}{\partial_s\varphi(s,x,y)}+q(x,y)\cdot\tilde{e}.$$

If f is a nonlinearity of "combustion" type satisfying (3.1.7), then the unique speed $c(e)$ that corresponds to problem (4.1.8) is given by

$$c(e)=\min_{\varphi\in E}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y), \tag{3.1.15}$$

$$c(e)\ =\max_{\varphi\in E}\ \inf_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y). \tag{3.1.16}$$

Furthermore, the min *in (3.1.15) and the* max *in (3.1.16) are attained by, and only by, the function $\phi(s,x,y)=u\left(\frac{s-x\cdot e}{c(e)},x,y\right)$ and its shifts $\phi(s+\tau,x,y)$ for any $\tau\in\mathbb{R}$, where u is the solution of (4.1.8) with a speed $c(e)$ (whose existence and uniqueness up to a translation in t follow from Theorem 3.1.5).*

The second result is concerned with "ZFK" nonlinearities.

Theorem 3.1.9 *Under the same notations of Theorem 3.1.8, if f is a nonlinearity of "ZFK" type satisfying (3.1.5-3.1.6) and (3.1.8), then the minimal speed $c^*_{\Omega,A,q,f}(e)$ is given by*

$$c^*_{\Omega,A,q,f}(e)=\min_{\varphi\in E}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y). \tag{3.1.17}$$

Furthermore, the min *is attained by the function $\phi^*(s,x,y)=u^*\left(\frac{s-x\cdot e}{c^*(e)},x,y\right)$ and its shifts $\phi^*(s+\tau,x,y)$ for any $\tau\in\mathbb{R}$, where u^* is any solution of (4.1.8) propagating with the speed $c^*(e)=c^*_{\Omega,A,q,f}(e)$.*

In particular, Theorem 3.1.9 yields that formula (3.1.17) holds in the "KPP" case (3.1.10) as well.

Remark 3.1.10 *In Theorem 3.1.8, the* min *and the* max *are attained by, and only by, the pulsating front $\phi(s,x,y)$ and its shifts $\phi(s+\tau,x,y)$ for all $\tau\in\mathbb{R}$. In Theorem 3.1.9, the* min *is achieved by the front $\phi^*(s,x,y)$ with the speed $c^*(e)$ and all its shifts $\phi^*(s+\tau,x,y)$. Actually, if the pulsating front ϕ^* is unique up to shift, then ϕ^* and its shifts are the unique minimizers in formula (3.1.17). The uniqueness is known in the "KPP" case (see Hamel, Roques [8]), but it is still open in the general "ZFK" case.*

We mention that a max-min formula of the type (3.1.16) can not hold for the minimal speed $c^*(e)$ in the "ZFK" or the "KPP" case. A simple justification is given in section 3.2.

The variational formulations of the speeds of propagation which are given in Theorems 3.1.8 and 3.1.9 are more general than those in Hamel [7] and Heinze, Papanicolaou, Stevens [10]. In Theorems 3.1.8 and 3.1.9, we consider nonhomogeneous nonlinearities $f=f(x,y,u)$ and the domain Ω is in the most general periodic situation. However, in [7], the domain was an infinite cylinder of $\mathbb{R}^N$ and the advection q was in the form of shear flows. Moreover, in this paper, the nonhomogeneous operator $\nabla\cdot(A\nabla u)$ replaces the Laplace operator Δu taken in [7]. On the other hand, in [10], the domain Ω was an infinite cylinder in $\mathbb{R}^N$ with a bounded cross section. Namely, $\Omega=\mathbb{R}\times\omega\subset\mathbb{R}^N$ where the cross section ω is a bounded domain in $\mathbb{R}^{N-1}$. Moreover, the authors did not consider an advection field in [10]. Finally, concerning the nonlinearities, they were depending only on u (i.e $f=f(u)$ and is satisfying either (3.1.11) or (3.1.12)) in both of [7] and [10].

Besides the fact that we consider here a wider family of diffusion and reaction coefficients, our assumptions are less strict than those supposed in [10] and [17]. Roughly speaking, the authors, in [10] and [17], assume a stability condition on the pulsating travelling fronts. We mention that such a stability condition is fulfilled in the homogenous setting; however, it has not been rigorously proved so far that this condition is satisfied in the heterogenous setting. Meanwhile, the assumptions of the present paper only involve the coefficients of the reaction-advection-diffusion equation (4.1.8), and they can then be checked easily.

Actually, in the "KPP" case, another "simpler" variational formula for the minimal speed $c^*(e)=c^*_{\Omega,A,q,f}(e)$ is known. This known formula involves only the linearized nonlinearity f at $u=0$. Namely, it follows from [2] that

Theorem 3.1.11 (Berestycki, Hamel, Nadirashvili [2]) *Let e be a fixed unit vector in $\mathbb{R}^d$ and let $\tilde{e}=(e,0,\ldots,0)\in\mathbb{R}^N$. Assume that f is a "KPP" nonlinearity and that Ω, A and q satisfy (3.1.2), (3.1.3) and (3.1.4) respectively. Then, the minimal speed $c^*(e)$ of pulsating fronts solving (4.1.8) and propagating in the direction of e is* *given by*

$$c^*(e) = c^*_{\Omega,A,q,f}(e) \; = \; \min_{\lambda>0} \frac{k(\lambda)}{\lambda}, \tag{3.1.18}$$

where $k(\lambda) = k_{\Omega,e,A,q,\zeta}(\lambda)$ *is the principal eigenvalue of the operator* $L_{\Omega,e,A,q,\zeta,\lambda}$ *which is defined by*

$$\begin{array}{rcl} L_{\Omega,e,A,q,\zeta,\lambda}\psi & := & \nabla\cdot(A\nabla\psi) \; + 2\lambda\tilde{e}\cdot A\nabla\psi \; + q\cdot\nabla\psi \\ & & +[\lambda^2\tilde{e}A\tilde{e} + \lambda\nabla\cdot(A\tilde{e}) + \lambda q\cdot\tilde{e} + \zeta]\psi \end{array} \tag{3.1.19}$$

acting on the set

$$\widetilde{E_\lambda} = \left\{\psi \in C^2(\overline{\Omega}), \psi \textit{ is L-periodic with respect to } x \textit{ and } \nu\cdot A\nabla\psi \; = \; -\lambda(\nu\cdot A\tilde{e})\psi \textit{ on } \partial\Omega\right\}.$$

3.2 Main tools: change of variables and maximum principles

In this section, we introduce some tools that will be used in different places of this paper in order to prove the main results.

Throughout this paper, $\tilde{e}$ will denote the vector in $\mathbb{R}^N$ defined by

$$\tilde{e} = (e, 0, \cdots, 0) = (e^1, \cdots, e^d, 0, \cdots, 0),$$

where $e^1, \cdots, e^d$ are the components of the vector e.

Our study is concerned with the model (4.1.8). Having a "combustion", a "ZFK", or a "KPP" nonlinearity, together with the assumptions (3.1.3) and (3.1.4), problem (4.1.8) has at least a classical solution (c, u) such that $c > 0$ and $u_t > 0$ (see Theorems 3.1.5 and 3.1.6). The function u is globally $C^{1,\mu}(\mathbb{R}\times\overline{\Omega})$ and $C^{2,\mu}$ with respect to (x, y) variables (for every $\mu \in [0, 1)$). It follows that $\nabla_{x,y}.(A\nabla u) \in C(\mathbb{R}\times\overline{\Omega})$. Having a unit direction $e \in \mathbb{R}^d$, and having a bounded classical solution (c, u) of (4.1.8) with $c = c(e)$ (combustion case) or $c \geq c^*(e)$ (ZFK or KPP case), we make the same change of variables as Xin [20]. Namely, let $\phi = \phi(s, x, y)$ be the function defined by

$$\phi(s, x, y) = u\left(\frac{s - x\cdot e}{c}, x, y\right) \text{ for all } s \in \mathbb{R} \text{ and } (x, y) \in \overline{\Omega}. \tag{3.2.1}$$

One then has

$$\forall (s,x,y)\in\mathbb{R}\times\overline{\Omega},\quad \left[\nabla_{x,y}\cdot(A\nabla_{x,y}\phi)+(\tilde{e}A\tilde{e})\phi_{ss}+\nabla_{x,y}\cdot(A\tilde{e}\phi_s)+\partial_s(\tilde{e}A\nabla_{x,y}\phi)\right](s,x,y)$$

$$=\nabla_{x,y}\cdot(A\nabla u)(t,x,y),$$

where $s=x\cdot e+ct$. Consequently,

$$F[\phi](s,x,y)=\ \nabla_{x,y}\cdot(A\nabla_{x,y}\phi)+(\tilde{e}A\tilde{e})\phi_{ss}+\nabla_{x,y}\cdot(A\tilde{e}\phi_s)+\partial_s(\tilde{e}A\nabla_{x,y}\phi)$$

is defined at each point $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$ and the map $(s,x,y)\mapsto F[\phi](s,x,y)$ belongs to $C(\mathbb{R}\times\overline{\Omega})$.

In all this paper, $L=L_c$ will denote the operator acting on the set E (given in Theorem 3.1.8) and which is defined by

$$\begin{aligned}\forall\,\varphi\in E,\ L\varphi\ &=\ \nabla_{x,y}\cdot(A\nabla_{x,y}\varphi)+(\tilde{e}A\tilde{e})\varphi_{ss}+\nabla_{x,y}\cdot(A\tilde{e}\varphi_s)+\partial_s(\tilde{e}A\nabla_{x,y}\varphi)\\ &\quad +q\cdot\nabla_{x,y}\varphi+(q\cdot\tilde{e}-c)\varphi_s\ \text{ in } C(\mathbb{R}\times\overline{\Omega})\\ &=\ F[\varphi]+q\cdot\nabla_{x,y}\varphi+(q\cdot\tilde{e}-c)\varphi_s\ \text{ in } C(\mathbb{R}\times\overline{\Omega}).\end{aligned}\tag{3.2.2}$$

It follows from above that if $\phi=\phi(s,x,y)$ is a function that is given by a pulsating travelling (c,u) solving (4.1.8) (under the change of variables (3.2.1)), then $F[\phi]\in C(\mathbb{R}\times\overline{\Omega})$, ϕ is globally bounded in $C^{1,\mu}(\mathbb{R}\times\overline{\Omega})$ (for every $\mu\in[0,1)$) and it satisfies the following degenerate elliptic equation

$$\begin{aligned}L\phi(s,x,y)+f(x,y,\phi)\ &=F[\phi](s,x,y)+\ q\cdot\nabla_{x,y}\phi(s,x,y)\\ &\quad+(q\cdot\tilde{e}-c)\phi_s(s,x,y)+f(x,y,\phi)\ =0\ \text{ in }\ \mathbb{R}\times\overline{\Omega},\end{aligned}\tag{3.2.3}$$

together with the boundary and periodicity conditions

$$\begin{cases}\phi \text{ is } L\text{-periodic with respect to } x,\\ \nu\cdot A(\nabla_{x,y}\phi+\tilde{e}\phi_s)=0 \text{ on } \mathbb{R}\times\overline{\Omega}.\end{cases}\tag{3.2.4}$$

Moreover, since $u(t,x,y)\to 0$ as $x\cdot e\to-\infty$ and $u(t,x,y)\to 1$ as $x\cdot e\to+\infty$ locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e, and since ϕ is $L-$periodic with respect to x, the change of variables $s=x\cdot e+ct$ guarantees that

$$\phi(-\infty,.,.)=0 \text{ and } \phi(+\infty,.,.)=1 \text{ uniformly in } (x,y)\in\overline{\Omega}.\tag{3.2.5}$$

Therefore, one can conclude that $\phi\in E$.

Remark 3.2.1 *It is now clear that a max-min formula of the type (3.1.16) can not* *hold for the minimal speed* $c^*(e) > 0$ *in the "ZFK" or the "KPP" case. Indeed, for each speed* $c \geq c^*(e)$*, there is a solution* (c, u) *of (4.1.8) such that* $u_t > 0$*, which gives birth to a function* $\phi = \phi(s, x, y)$ *under the change of variables (3.2.1). Owing to the above discussions the function* $\phi \in E$ *and it satisfies*

$$c = R\phi(s, x, y) \quad \text{for all} \quad (s, x, y) \in \mathbb{R} \times \overline{\Omega}.$$

Therefore

$$\sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y) \geq c.$$

Since one can choose any $c \geq c^*(e)$*, one concludes that*

$$\sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y) = +\infty$$

in the "ZFK" or the "KPP" case.

Remark 3.2.2 (The same formulæ of Theorems 3.1.8 and 3.1.9, but over a subset of E**)**
If the restriction of the nonlinear source f *in (4.1.8) is* $C^{1,\delta}(\overline{\Omega} \times [0, 1])$*, one can then conclude that (see the proof of Proposition 6.3 in [1]) any solution* u *of (4.1.8) satisfies:*

$$\forall (t, x, y) \in \mathbb{R} \times \overline{\Omega}, \quad |\partial_{tt} u(t, x, y)| \leq M\, \partial_t u(t, x, y) \tag{3.2.6}$$

for some constant M *independent of* (t, x, y)*. In other words, the function*

$$\phi(s, x, y) = u((s - x \cdot e)/c, x, y)$$

(where $c = c(e)$ *in the "combustion" case, and* $c = c^*(e)$ *in the "ZFK" or the "KPP" case) satisfies*

$$\forall (s, x, y) \in \mathbb{R} \times \overline{\Omega}, \quad |\partial_{ss} \phi(s, x, y)| \leq (M/c)\, \partial_s \phi(s, x, y).$$

Let E' *be the functional subset of* E *defined by*

$$E' = \left\{ \varphi \in \ E,\ \exists C > 0,\ |\partial_{ss} \varphi(s, x, y)| \leq C\, \partial_s \varphi(s, x, y) \ \ \text{for all } (s, x, y) \in \mathbb{R} \times \overline{\Omega} \right\}.$$

The previous facts together with the discussions at the beginning of this section imply that the functions ϕ *and* ϕ^* *of Theorems 3.1.8 and 3.1.9 are elements of* $E' \subset E$*. These theorems also yield that the* max-min *and the* min-max *formulæ can also hold over the* *subset* $E^{'}$ *of* E.

Namely, in the case of a "combustion" nonlinearity

$$c(e) = \min_{\varphi \in E^{'}} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y) \tag{3.2.7}$$

and

$$c(e) = \max_{\varphi \in E^{'}} \inf_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y). \tag{3.2.8}$$

Moreover, the min *and the* max *are attained at, and only at, the function* $\phi(s,x,y)$ *and its shifts* $\phi(s+\tau,x,y)$ *for any* $\tau\in\mathbb{R}$.

On the other hand, only a min-max *formula holds in the case of "ZFK" or "KPP" nonlinearities. That is*

$$c^*(e) = \min_{\varphi \in E^{'}} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s,x,y). \tag{3.2.9}$$

Moreover, the min *is attained at the function* $\phi^*(s,x,y)$ *and its shifts* $\phi^*(s+\tau,x,y)$ *for any* $\tau\in\mathbb{R}$.

In the proofs of the variational formulæ which were given in Theorem 3.1.8 and Theorem 3.1.9, we will use two versions of the maximum principle in unbounded domains for some problems related to (3.2.2-3.2.4) and (3.2.5). Such generalized maximum principles were proved in Berestycki, Hamel [1] in a slightly more general framework:

Lemma 3.2.3 ([1]) *Let* e *be a fixed unit vector in* $\mathbb{R}^d$. *Let* $g(x,y,u)$ *be a globally bounded and globally Lipschitz-continuous function defined in* $\overline{\Omega}\times\mathbb{R}$ *and assume that* g *is non-increasing with respect to* u *in* $\overline{\Omega}\times(-\infty,\delta]$ *for some* $\delta>0$. *Let* $h\in\mathbb{R}$ *and* $\Sigma_h^- := (-\infty,h)\times\Omega$. *Let* $c\neq 0$ *and* $\phi^1(s,x,y)$, $\phi^2(s,x,y)$ *be two bounded and globally* $C^{1,\mu}\left(\overline{\Sigma_h^-}\right)$ *functions (for some* $\mu>0$*) such that*

$$\left\{\begin{array}{rcl} L\,\phi^1 + g(x,y,\phi^1) & \geq & 0 \;\; in \;\; \mathcal{D}'(\Sigma_h^-),\\ L\,\phi^2 + g(x,y,\phi^2) & \leq & 0 \;\; in \;\; \mathcal{D}'(\Sigma_h^-),\\ \nu\cdot A\,[\tilde{e}(\phi_s^1-\phi_s^2)+\nabla_{x,y}(\phi^1-\phi^2)] & \leq & 0 \;\; on \; (-\infty,h]\times\partial\Omega,\\ \lim\limits_{s_0\to-\infty}\sup\limits_{\{s\leq s_0,\;(x,y)\in\overline{\Omega}\}}[\phi^1(s,x,y)-\phi^2(s,x,y)] & \leq & 0, \end{array}\right. \tag{3.2.10}$$

where

$$\begin{array}{rcl} L\,\phi & := & \nabla_{x,y}\cdot(A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \nabla_{x,y}\cdot(A\tilde{e}\phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi)\\ & & +\,q\cdot\nabla_{x,y}\phi \,+\, (q\cdot\tilde{e}\,-\,c)\phi_s, \end{array} \tag{3.2.11}$$

and $\tilde{e}$ *denotes the vector* $(e, 0, \cdots, 0) \in \mathbb{R}^N$.

If $\phi^1 \le \delta$ *in* $\overline{\Sigma_h^-}$ *and* $\phi^1(h, x, y) \le \phi^2(h, x, y)$ *for all* $(x, y) \in \overline{\Omega}$, *then*

$$\phi^1 \le \phi^2 \quad in\ \overline{\Sigma_h^-}.$$

Remark 3.2.4 *Note here that* ϕ^1, ϕ^2, q, A *and* g *are not assumed to be* $L-$*periodic in* x *and that* q *is not assumed to satisfy (3.1.4).*

Proof. Since ϕ^1 and ϕ^2 are globally bounded, one has $\phi^1 - \varepsilon \le \phi^2$ in $\overline{\Sigma_h^-}$ for $\varepsilon > 0$ large enough. Let us set

$$\varepsilon^* = \inf\left\{\varepsilon > 0,\ \phi^1 - \varepsilon \le \phi^2 \text{ in } \overline{\Sigma_h^-}\right\} \ge 0.$$

By continuity, one has $\phi^1 - \varepsilon^* \le \phi^2$ in $\overline{\Sigma_h^-}$. Thus, to complete the proof of Lemma 3.2.3, it suffices to to prove that $\varepsilon^* = 0$.

Assume $\varepsilon^* > 0$. There exist a sequence $\{\varepsilon_n\}$ converging to ε^*, with $0 < \varepsilon_n < \varepsilon$ for all n, and a sequence of points $(s_n, x_n, y_n) \in \overline{\Sigma_h^-}$ such that

$$\phi^1(s_n, x_n, y_n) - \varepsilon_n \ge \phi^2(s_n, x_n, y_n).$$

Owing to the assumption that $\lim_{s_0 \to -\infty} \sup_{\{s \le s_0,\ (x,y) \in \overline{\Omega}\}} [\phi^1(s, x, y) - \phi^2(s, x, y)] \le 0$, the sequence (s_n) is bounded from below (if not, there exists a subsequence denoted by (s_n) such that $s_n \to -\infty$ as $n \to +\infty$. One consequently has

$$\begin{aligned} 0 &\ge \lim_{s_n \to -\infty} \sup_{\{s \le s_n,\ (x,y) \in \overline{\Omega}\}} [\phi^1(s_n, x, y) - \phi^2(s_n, x, y)] \\ &\ge \lim_{n \to +\infty} [\phi^1(s_n, x_n, y_n) - \phi^2(s_n, x_n, y_n)] \\ &\ge \lim_{n \to +\infty} \varepsilon_n \\ &= \varepsilon^*. \end{aligned}$$

It follows that $\varepsilon^* = 0$ and this contradicts with the assumption that $\varepsilon^* > 0$). Moreover, the sequence (s_n) is also bounded from above ($s_n \le h$) and hence, one can assume, up to extraction of some subsequence, that $s_n \to \bar{s} \in (-\infty, h]$ as $n \to +\infty$. On the other hand, let $\tilde{x}_n$ be in $\prod_{i=1}^{d} L_i\mathbb{Z}$ such that $(x_n - \tilde{x}_n, y_n) \in \overline{C}$. Up to extraction of some subsequence, one can also assume that $(x_n - \tilde{x}_n, y_n) \to (\bar{x}, \bar{y}) \in \overline{C}$ as $n \to +\infty$.

Set

$$\phi_n^i(s, x, y) = \phi^i(s, x + \tilde{x}_n, y) \text{ for all } (s, x, y) \in \overline{\Sigma_h^-},\ i = 1, 2.$$

The above functions are defined in the same set $\overline{\Sigma_h^-} = (-\infty, h] \times \overline{\Omega}$ because of the choice of the $\tilde{x}_n$ and the $L-$periodicity of Ω with respect to x. From the regularity assumptions for ϕ^1 and ϕ^2 and up to extraction of some subsequence, the functions ϕ_n^i converge, in C^1_{loc}, to two functions $\phi_\infty^i \in C^{1,\mu}(\overline{\Sigma_h^-})$. Similarly, since q and A are globally $C^{\overline{\Omega}}$ (q and A are $C^{1,\delta}(\overline{\Omega})$ and $C^{2,\delta}(\overline{\Omega})$ respectively), one can assume that the fields $q_n(x,y) = q(x+\tilde{x}_n, y)$ and $A_n(x,y) = A(x+\tilde{x}_n, y)$ converge locally in $\overline{\Omega}$ to two globally bounded and Lipschitz fields q_∞ and A_∞ as $n \to +\infty$. The matrix satisfies the same ellipticity condition (given in (3.1.3)) as A.

The functions ϕ_n^i satisfy

$$L_n\, \phi_n^1 - L_n\, \phi_n^2 \geq -g(x+\tilde{x}_n, y, \phi_n^1(s,x,y)) + g(x+\tilde{x}_n, y, \phi_n^2(s,x,y)) \quad \text{in } D'(\Sigma_h^-),$$

where

$$\begin{aligned} L_n\, \phi \;\; := \;\; & \nabla_{x,y} \cdot (A_n \nabla_{x,y} \phi) + (\tilde{e} A_n \tilde{e}) \phi_{ss} + \nabla_{x,y} \cdot (A_n \tilde{e} \phi_s) + \partial_s (\tilde{e} A_n \nabla_{x,y} \phi) \\ & + q_n \cdot \nabla_{x,y} \phi \, + \, (q_n \cdot \tilde{e} \, - \, c) \phi_s. \end{aligned}$$

Since $\phi^1 \leq \delta$ in $\overline{\Sigma_h^-}$ and $g(x,y,u)$ is non-increasing with respect to u in $\overline{\Omega} \times (-\infty, \delta]$, one gets

$$L_n\, \phi_n^1 - L_n\, \phi_n^2 \geq -g(x+\tilde{x}_n, y, \phi_n^1(s,x,y) - \varepsilon^*) + g(x+\tilde{x}_n, y, \phi_n^2(s,x,y)) \text{ in } \mathcal{D}'(\Sigma_h^-). \tag{3.2.12}$$

From the assumptions of Lemma 3.2.3, one can also assume, up to extraction of some subsequence, that the functions

$$R_n(s,x,y) := -g(x+\tilde{x}_n, y, \phi_n^1(s,x,y) - \varepsilon^*) + g(x+\tilde{x}_n, y, \phi_n^2(s,x,y))$$

converge to a function $R_\infty(s,x,y)$ locally in $\overline{\Sigma_h^-}$. Since $|R_n| \leq ||g||_{Lip} |\phi_n^1 - \varepsilon^* - \phi_n^2|$ for all n, one gets $|R_\infty| \leq ||g||_{Lip} |\phi_\infty^1 - \varepsilon^* - \phi_\infty^2|$ at the limit. In other words, there exists a globally bounded function $B(s,x,y)$ defined in $\overline{\Sigma_h^-}$ such that

$$R_\infty(s,x,y) = B(s,x,y) \left[\phi_\infty^1(s,x,y) - \varepsilon^* - \phi_\infty^2(s,x,y) \right] \quad \text{for all } (s,x,y) \in \overline{\Sigma_h^-}.$$

By passing to the limit in (3.2.12), it follows that the functions ϕ_∞^1 and ϕ_∞^2 satisfy

$$L_\infty \phi_\infty^1 - L_\infty \phi_\infty^2 \geq B(s,x,y)(\phi_\infty^1 - \varepsilon^* - \phi_\infty^2) \quad \text{in } \mathcal{D}'(\Sigma_h^-)$$

where

$$\begin{array}{rcl} L_\infty \phi & := & \nabla_{x,y}\cdot(A_\infty\nabla_{x,y}\phi)+(\tilde{e}A_\infty\tilde{e})\phi_{ss}+\nabla_{x,y}\cdot(A_\infty\tilde{e}\phi_s)+\partial_s(\tilde{e}A_\infty\nabla_{x,y}\phi)\\ & & +\,q_\infty\cdot\nabla_{x,y}\phi\,+\,(q_\infty\cdot\tilde{e}\,-\,c)\phi_s. \end{array}$$

Moreover, the inequalities $\phi^1_\infty \le \delta$ in $\overline{\Sigma_h^-}$ and $\phi^1_\infty(h,.,.) \le \phi^2_\infty(h,.,.)$ in $\overline{\Omega}$ hold as well. Furthermore, $\phi^1_\infty - \varepsilon^* \le \phi^2_\infty$ and

$$\forall\, n, \quad \phi^1(s_n,x_n,y_n)-\varepsilon_n > \phi^2(s_n,x_n,y_n),$$

hence

$$\forall\, n, \quad \phi^1_n(s_n,x_n-\tilde{x}_n,y_n)-\varepsilon_n > \phi^2_n(s_n,x_n-\tilde{x}_n,y_n).$$

Passing to the limit as $n\to+\infty$, one gets $\phi^1_\infty(\bar{s},\bar{x},\bar{y})-\varepsilon^* \ge \phi^2_\infty(\bar{s},\bar{x},\bar{y})$. Therefore,

$$\phi^1_\infty(\bar{s},\bar{x},\bar{y})-\varepsilon^* = \phi^2_\infty(\bar{s},\bar{x},\bar{y}),$$

whence $\bar{s}<h$.

Coming back to the variables (t,x,y), let us define

$$E_h=\{(t,x,y)\in\mathbb{R}\times\Omega,\ ct+x\cdot e<h\}$$

and set

$$u^i(t,x,y)=\phi^i_\infty(ct+x\cdot e,x,y)\quad\text{for all } (t,x,y)\in\overline{E_h},\ i=1,2.$$

The function $z:=u^1-\varepsilon^*-u^2$, which is defined and globally C^1 in $\overline{E_h}$, satisfies

$$\nabla_{x,y}\cdot(A_\infty\nabla_{x,y}z)-q_\infty(x,y)\cdot\nabla_{x,y}z-\partial_t z\ge b(t,x,y)z\quad\text{in } \mathcal{D}'(E_h)$$

where the function $b(t,x,y):=B(ct+x\cdot e,x,y)$ is globally bounded in $\overline{E_h}$. Moreover,

$$\nu\cdot\ A_\infty\nabla_{x,y}z\le 0\quad\text{on } \{ct+x\cdot e\le h,\ (x,y)\in\partial\Omega\}.$$

On the other hand, the function z is non-positive and it vanishes at the point $(\bar{t},\bar{x},\bar{y})=\left(\dfrac{\bar{s}-\bar{x}\cdot e}{c},\bar{x},\bar{y}\right)$, which is such that $c\,\bar{t}+\bar{x}\cdot e(=\bar{s})<h$. Therefore, it follows from the maximum principle that $z\equiv 0$ in $\overline{E_h}\cap\{t\le\bar{t}\}$. In other words, $u^1-\varepsilon^*=u^2$ in $\overline{E_h}\cap\{t\le\bar{t}\}$. In particular, one has

$$\phi^1_\infty-\varepsilon^*=\phi^2_\infty\quad\text{ in } \overline{\Sigma_h^-}\cap\{\frac{s-x\cdot e}{c}\le\bar{t}\}.$$

Since the set $\{x \cdot e\}$ is not bounded from above or below, there exists a point $(h, x_0, y_0) \in \overline{\Sigma_h^-} \cap \{\frac{s-x\cdot e}{c} \leq \bar{t}\}$. At that point, one has $\phi^1_\infty(h, x_0, y_0) - \varepsilon^* = \phi^2_\infty(h, x_0, y_0)$. But the later is impossible because $\phi^1_\infty \leq \phi^2_\infty$ for $s = h$.

Hence, the assumption $\varepsilon^* > 0$ is ruled out and the proof of Lemma 3.2.3 is complete. □

Changing $\phi^1(s, x, y)$, $\phi^2(s, x, y)$ and $g(x, y, s)$ into $1 - \phi^1(-s, x, y)$, $1 - \phi^2(-s, x, y)$ and $-g(x, y, 1 - s)$ respectively in Lemma 3.2.3 leads to the following

Lemma 3.2.5 ([1]) *Let e be a fixed unit vector in $\mathbb{R}^d$. Let $g(x, y, u)$ be a globally bounded and globally Lipschitz-continuous function defined in $\overline{\Omega} \times \mathbb{R}$ and assume that g is non-increasing with respect to u in $\overline{\Omega} \times [1 - \delta, +\infty)$ for some $\delta > 0$. Let $h \in \mathbb{R}$ and $\Sigma_h^+ := (h, +\infty) \times \Omega$. Let $c \neq 0$ and $\phi^1(s, x, y)$, $\phi^2(s, x, y)$ be two bounded and globally $C^{1,\mu}\left(\overline{\Sigma_h^+}\right)$ functions (for some $\mu > 0$) such that*

$$\begin{cases} L\,\phi^1 + g(x, y, \phi^1) \geq 0 \quad in \ \mathcal{D}'(\Sigma_h^+), \\ L\,\phi^2 + g(x, y, \phi^2) \leq 0 \quad in \ \mathcal{D}'(\Sigma_h^+), \\ \nu \cdot A\,[\tilde{e}(\phi^1_s - \phi^2_s) + \nabla_{x,y}(\phi^1 - \phi^2)] \leq 0 \ on \ [h, +\infty) \times \partial\Omega, \\ \lim\limits_{s_0 \to +\infty} \sup\limits_{\{s \geq s_0,\ (x,y) \in \overline{\Omega}\}} [\phi^1(s, x, y) - \phi^2(s, x, y)] \leq 0, \end{cases} \tag{3.2.13}$$

where L is the same operator as in Lemma 3.2.3.

If $\phi^2 \geq 1 - \delta$ in $\overline{\Sigma_h^+}$ and $\phi^1(h, x, y) \leq \phi^2(h, x, y)$ for all $(x, y) \in \overline{\Omega}$, then

$$\phi^1 \leq \phi^2 \quad in \ \ \overline{\Sigma_h^+}.$$

3.3 Case of a "combustion" nonlinearity

This section is devoted to prove Theorem 3.1.8, where the nonlinearity f satisfies the assumptions (3.1.5-3.1.6) and (3.1.7).

3.3.1 Proof of formula (3.1.15)

Having a prefixed unit direction $e \in \mathbb{R}^d$, and since the coefficients A and q of problem (4.1.8) satisfy the assumptions (3.1.3) and (3.1.4), it follows, from Theorem 3.1.5, that there exists a unique pulsating travelling front $(c(e), u)$ (u is unique up to a translation in the time variable) which solves problem (4.1.8). Moreover, $\partial_t u > 0$ in $\mathbb{R} \times \overline{\Omega}$. We will complete the proof of (3.1.15) via two steps.

Step 1. After the discussions done in the section 3.2, the existence of a classical solution $(c(e), u)$, satisfying (4.1.8), implies the existence of a globally $C^1(\mathbb{R}\times\overline{\Omega})$ function $\phi(s, x, y)$ satisfying $0 \leq \phi \leq 1$ in $\mathbb{R} \times \overline{\Omega}$, with

$$\begin{cases} \phi \text{ is } L\text{−periodic with respect to } x, \\ L\phi(s, x, y) + f(x, y, \phi) = 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \overline{\Omega}), \\ \nu \cdot A(\nabla_{x,y}\phi + \tilde{e}\phi_s) = 0 \text{ in } \mathbb{R} \times \partial\Omega, \\ \phi(-\infty, ., .) = 0, \text{ and } \phi(+\infty, ., .) = 1 \text{ uniformly in } (x, y) \in \overline{\Omega}, \end{cases} \tag{3.3.1}$$

where L is the operator defined in (3.2.2) for $c = c(e)$. We also recall that the two functions u and ϕ satisfy the relation

$$u(t, x, y) = \phi(x \cdot e + c(e)t, x, y), \quad (t, x, y) \in \mathbb{R} \times \overline{\Omega}.$$

One has $\partial_s\phi > 0$ in $\mathbb{R} \times \overline{\Omega}$ and this is equivalent to say that the function $u = u(t, x, y)$ is increasing in t, since $c(e) > 0$.

Together with the facts in section 3.2.1, one gets that the function $\phi \in E$. Furthermore, (3.3.1) yields that

$$\forall\, s \in \mathbb{R},\ \forall (x, y) \in \overline{\Omega},\ c(e) = R\,\phi(s, x, y), \tag{3.3.2}$$

and

$$L\phi(s, x, y) + f(x, y, \phi) = 0, \tag{3.3.3}$$

where $R\phi$ is the function defined in Theorem 3.1.8. In other words, the L−periodic (with respect to x) function $R\phi$ is constant over $\mathbb{R} \times \overline{\Omega}$ and it is equal to $c(e)$.

It follows, from (3.3.2) and from the above explanations, that

$$c(e) \geq \inf_{\varphi\in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s, x, y).$$

To complete the proof of formula (3.1.15), we assume that

$$c(e) > \inf_{\varphi\in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s, x, y).$$

Then, there exists a function $\psi = \psi(s, x, y) \in E$ such that

$$c(e) > \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\psi\,(s, x, y).$$

Since the function $\psi \in E$, one then has $\psi_s(s,x,y) > 0$ for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$. This yields that

$$L\psi(s,x,y) + f(x,y,\psi) < 0 \text{ in } \mathbb{R} \times \overline{\Omega}, \tag{3.3.4}$$

where L is the operator defined in (3.2.2) for $c = c(e)$.

Notice that the later holds for each function of the type

$$\psi^\tau(s,x,y) := \psi(s+\tau,x,y)$$

because of the invariance of (3.3.4) with respect to s and because the advection field q and the diffusion matrix A depend on the variables (x,y) only. That is

$$L\psi^\tau(s,x,y) + f(x,y,\psi^\tau) < 0 \text{ in } \mathbb{R} \times \overline{\Omega}. \tag{3.3.5}$$

Step 2. In order to draw a contradiction, we are going to slide the function ψ with respect to ϕ. From the limiting conditions satisfied by these two functions, there exists a real number $B > 0$ such that

$$\begin{cases} \phi(s,x,y) \leq \theta & \text{for all } s \leq -B, \ (x,y) \in \overline{\Omega}, \\ \psi(s,x,y) \geq 1-\rho & \text{for all } s \geq B, \ (x,y) \in \overline{\Omega}, \end{cases}$$

and

$$\phi(B,x,y) \geq 1-\rho \ \text{ for all } (x,y) \in \overline{\Omega}, \tag{3.3.6}$$

where θ and ρ are the values that appear in the conditions (3.1.7) satisfied by the "combustion" nonlinearity f. Taking $\tau \geq 2B$, and since ψ is increasing with respect to s, one gets that $\phi(-B,x,y) \leq \psi^\tau(-B,x,y)$ for all $(x,y) \in \overline{\Omega}$ and $\psi^\tau \geq 1-\rho$ in $\overline{\Sigma^+_{-B}}$.

It follows from Lemma 3.2.3 (take $\delta = \theta, \ h = -B, \ \phi^1 = \phi,$ and $\phi^2 = \psi^\tau$) that $\phi \leq \psi^\tau$ in $\overline{\Sigma^-_{-B}}$. Moreover, Lemma 3.2.5 (take $\delta = \rho, \ h = -B, \ \phi^1 = \phi,$ and $\phi^2 = \psi^\tau$) implies that $\phi \leq \psi^\tau$ in $\overline{\Sigma^+_{-B}}$. Consequently, $\phi \leq \psi^\tau$ in $\mathbb{R} \times \overline{\Omega}$ for all $\tau \geq 2B$.

Let us now decrease τ and set

$$\tau^* = \inf\{\tau \in \mathbb{R}, \quad \phi \leq \psi^\tau \text{ in } \mathbb{R} \times \overline{\Omega} \ \}.$$

First one notes that $\tau^* \leq 2B$. On the other hand, the limiting conditions $\psi(-\infty,.,.) = 0$ and $\phi(+\infty,.,.) = 1$ imply that τ^* is finite. By continuity, $\phi \leq \psi^{\tau^*}$ in $\mathbb{R} \times \overline{\Omega}$. Two cases may occur according to the value of $\sup_{[-B,B]\times\overline{\Omega}} \left(\phi - \psi^{\tau^*}\right)$.

<u>*case 1:*</u> suppose that

$$\sup_{[-B,B]\times\overline{\Omega}} \left(\phi - \psi^{\tau^*}\right) < 0.$$

Since the functions ψ and ϕ are globally $C^1(\mathbb{R}\times\overline{\Omega})$ there exists $\eta>0$ such that the above inequality holds for all $\tau\in[\tau^*-\eta,\tau^*]$. Choosing any τ in the interval $[\tau^*-\eta,\tau^*]$, and applying Lemma 3.2.3 to the functions ψ^τ and ϕ, one gets that

$$\phi(s,x,y)\leq\psi^\tau(s,x,y)\ \text{ for all } s\leq -B,\ (x,y)\in\overline{\Omega},$$

together with the inequality

$$\phi(s,x,y)<\psi^\tau(s,x,y)\ \text{ for all } s\in[-B,B],\ \text{and for all } (x,y)\in\overline{\Omega}.$$

Owing to (3.3.6) and to the above inequality, it follows that

$$\psi^\tau(B,x,y)\geq 1-\rho \text{ in } \overline{\Omega}.$$

Moreover, since the function ψ is increasing in s, one gets that $\psi^\tau\geq 1-\rho$ in $\overline{\Sigma_B^+}$. Lemma 3.2.5, applied to ϕ and ψ^τ in $\overline{\Sigma_B^+}$, yields that

$$\phi(s,x,y)\leq\psi^\tau(s,x,y) \text{ for all } s\geq B,\ (x,y)\in\overline{\Omega}.$$

As a consequence, one obtains $\phi\leq\psi^\tau$ in $\mathbb{R}\times\overline{\Omega}$, and that contradicts the minimality of τ^*. Therefore, case 1 is ruled out.

case 2: suppose that

$$\sup_{[-B,B]\times\overline{\Omega}}\left(\phi-\psi^{\tau^*}\right)=0.$$

Then, there exists a sequence of points (s_n,x_n,y_n) in $[-B,B]\times\overline{\Omega}$ such that

$$\phi(s_n,x_n,y_n)-\psi^\tau(s_n,x_n,y_n)\to 0 \text{ as } n\to+\infty.$$

Due to the $L-$ periodicity of the functions ϕ and ψ, one can assume that $(x_n,y_n)\in\overline{C}$. Consequently, one can assume, up to extraction of a subsequence, that $(s_n,x_n,y_n)\to(\bar{s},\bar{x},\bar{y})\in[-B,B]\times\overline{C}$ as $n\to+\infty$. By continuity, one gets $\phi(\bar{s},\bar{x},\bar{y})=\psi^{\tau^*}(\bar{s},\bar{x},\bar{y})$.

We return now to the variables (t,x,y). Let

$$\begin{aligned} z(t,x,y) &= \phi(x\cdot e+c(e)\,t,x,y)-\psi(x\cdot e+c(e)\,t+\tau^*,x,y) \text{ for all } (t,x,y)\in\mathbb{R}\times\overline{\Omega},\\ &= u(t,x,y)-\psi(x\cdot e+c(e)\,t+\tau^*,x,y). \end{aligned}$$

Since the functions ϕ and ψ are in E, it follows that the function z is globally $C^1(\mathbb{R}\times\overline{\Omega})$

and it satisfies

$$\forall\,(t,x,y)\in\mathbb{R}\times\Omega,\quad \nabla_{x,y}\cdot(A\nabla z)(t,x,y)=F[\phi](s,x,y)-F[\psi^{\tau^*}](s,x,y),$$

where $s=x\cdot e+c(e)t$. Thus, $\nabla_{x,y}\cdot(A\nabla z)\in C(\mathbb{R}\times\overline{\Omega})$. Moreover, the function z is non positive and it vanishes at the point $((\bar{s}-\bar{x}\cdot e)/c(e),\bar{x},\bar{y})$. It satisfies the boundary condition $\nu\cdot(A\nabla z)=0$ on $\mathbb{R}\times\partial\Omega$. Furthermore, it follows, from (3.3.2) and (3.3.4), that

$$\partial_t z-\nabla_{x,y}\cdot(A\nabla z)+q(x,y)\cdot\nabla_{x,y}z\le f(x,y,\phi)-f(x,y,\psi^{\tau^*}).$$

However, the function f is globally Lipschitz-continuous in $\overline{\Omega}\times\mathbb{R}$; hence, there exists a bounded function $b(t,x,y)$ such that

$$\partial_t z-\nabla_{x,y}\cdot(A\nabla z)+q(x,y)\cdot\nabla_{x,y}z+b(t,x,y)\,z\le 0 \text{ in } \mathbb{R}\times\Omega,$$

with $z(t,x,y)\le 0$ for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$.

Applying the strong parabolic maximum principle and Hopf lemma, one gets that $z(t,x,y)=0$ for all $t\le(\bar{s}-\bar{x}\cdot e)/c(e)$ and for all $(x,y)\in\overline{\Omega}$. On the other hand, it follows from the definition of z and from the $L-$periodicity of the functions ϕ and ψ that $z(t,x,y)=0$ for all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$. Consequently,

$$\phi(s,x,y)=\psi^{\tau^*}(s,x,y)=\psi(s+\tau^*,x,y) \text{ for all } (s,x,y)\in\mathbb{R}\times\overline{\Omega}.$$

Referring to the equations (3.3.3) and (3.3.5), one gets a contradiction. Thus, case 2 is ruled out too, and that completes the proof of the formula (3.1.15).

Remark 3.3.1 (The uniqueness, up to a shift, of the minimizer in (3.1.15)) *If $\psi\in E$ is a minimizer in (3.1.15). The above arguments imply that case 2 necessarily occurs, and that ψ is equal to a shift of ϕ. In other words, the minimum in (3.1.15) is realized by and only by the shifts of ϕ.*

3.3.2 Proof of formula (3.1.16)

In this subsection, we are going to prove the "max-min" formula of the speed of propagation $c(e)$ whenever the nonlinearity f is of the "combustion" type. The tools and techniques which one uses here are similar to those used in the previous subsection. However, we are going to sketch the proof of formula (3.1.16) for the sake of completeness.

As it was justified in the previous subsection, one easily gets that

$$c(e) \leq \sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\varphi(s,x,y)$$

and

$$\forall\,(s,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad c(e) = R\phi(s,x,y),$$

where

$$\phi(s,x,y) = u\left((s - x \cdot e)/c(e), x, y\right), \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega},$$

and $u = u(t,x,y)$ is the unique (up to a translation in t) pulsating travelling front solving problem (4.1.8) and propagating in the speed $c(e)$. We recall that the function $\phi \in E$ (see section 3.2). It follows that the function ϕ satisfies the following

$$\begin{cases} \phi \text{ is } L\text{−periodic with respect to } x, \\ \phi \text{ is increasing in } s \in \mathbb{R}, \\ L\phi(s,x,y) + f(x,y,\phi) = 0 \text{ in } \mathbb{R} \times \overline{\Omega}, \\ \nu \cdot A(\nabla_{x,y}\phi + \tilde{e}\phi_s) = 0 \text{ in } \mathbb{R} \times \partial\Omega, \\ \phi(-\infty,.,.) = 0, \text{ and } \phi(+\infty,.,.) = 1 \text{ uniformly in } (x,y) \in \overline{\Omega}, \end{cases} \tag{3.3.7}$$

where L is the operator defined in (3.2.2) for $c = c(e)$.

Notice that the later holds also for each function of the type

$$\phi^{\tau}(s,x,y) := \phi(s+\tau,x,y)$$

because of the invariance of (3.3.8) with respect to s and because the advection field q and the diffusion matrix A depend on the variables (x,y) only.

To complete the proof of formula (3.1.16), we assume that

$$c(e) < \sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\varphi(s,x,y).$$

Hence, there exists $\psi \in E$ such that

$$c(e) < R\psi\,(s,x,y), \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Since the function $\psi \in E$, one then has $\psi_s(s,x,y) > 0$ for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$. This yields that

$$L\psi(s,x,y) + f(x,y,\psi) > 0 \text{ in } \mathbb{R} \times \overline{\Omega}. \tag{3.3.8}$$

To get a contradiction, we are going to slide the function ϕ with respect to ψ. In fact, the limiting conditions satisfied by ψ and ϕ, which are elements of E, yield that there exists a real positive number B such that

$$\begin{cases} \psi(s,x,y) \leq \theta & \text{for all } s \leq -B,\ (x,y) \in \overline{\Omega}, \\ \phi(s,x,y) \geq 1-\rho & \text{for all } s \geq B,\ (x,y) \in \overline{\Omega}, \end{cases}$$

and

$$\psi(B,x,y) \geq 1-\rho \quad \text{for all } (x,y) \in \overline{\Omega}, \tag{3.3.9}$$

where θ and ρ are the values appearing in the conditions (3.1.7) satisfied by the nonlinearity f. Having $\tau \geq 2B$, one applies Lemma 3.2.3 (taking $\delta = \theta,\ h = -B,\ \phi^1 = \psi,$ and $\phi^2 = \phi^\tau$) and Lemma 3.2.5 (taking $\delta = \rho,\ h = -B,\ \phi^1 = \psi,$ and $\phi^2 = \phi^\tau$) to the functions ϕ^τ and ψ, over the domains Σ^-_{-B} and Σ^+_{-B} respectively, to get that $\psi \leq \phi^\tau$ in Σ^-_{-B} and $\psi \leq \phi^\tau$ in Σ^+_{-B}. Consequently, one can conclude that

$$\forall\, \tau \geq 2B, \quad \psi \leq \phi^\tau \quad \text{in } \mathbb{R} \times \overline{\Omega}.$$

Let us now decrease τ and set

$$\tau^* = \inf\{\tau \in \mathbb{R}, \quad \psi \leq \phi^\tau \text{ in } \mathbb{R} \times \overline{\Omega}\ \}.$$

It follows, from the limiting conditions $\psi(+\infty,.,.) = 1$ and $\phi(-\infty,.,.) = 0$, that τ^* is finite. By continuity, we have $\psi \leq \phi^{\tau^*}$. In this situation, two cases may occur. Namely,

$$\text{case A:} \qquad \sup_{[-B,B]\times\overline{\Omega}} \left(\psi - \phi^{\tau^*}\right) < 0,$$

or

$$\text{case B:} \qquad \sup_{[-B,B]\times\overline{\Omega}} \left(\psi - \phi^{\tau^*}\right) = 0.$$

Imitating the ideas and the skills used in case 1 and case 2 during the proof of formula (3.1.15), one gets that case A (owing to minimality of τ^*) and case B (owing to (3.3.7) and (3.3.8)) are ruled out.

Therefore, the assumption that

$$c(e) < \sup_{\varphi \in E} \inf_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y)$$

is false, and that completes the proof of formula (3.1.16).

Remark 3.3.2 (The uniqueness, up to a shift, of the maximizer in (3.1.16))

Similar to what we have already mentioned in Remark 3.3.1, if $\psi \in E$ is a maximizer in (3.1.16), then the above arguments yield that case B necessarily occurs, and that ψ is equal to a shift of ϕ. One then concludes that the maximum in (3.1.16) is realized by, and only by, the shifts of ϕ.

3.4 Case of "ZFK" or "KPP" nonlinearities: proof of formula (3.1.17)

This section is devoted to the proof of Theorem 3.1.9. We assume that the nonlinear source f is of "ZFK" type. Remember that this case includes the class of "KPP" nonlinearities. In the first subsection, we give a unified proof including an argument of the "sliding method". In the second subsection, we give a "simpler" proof of the $\min - \max$ formula (3.1.17) under some additional assumption on the "ZFK" nonlinearity $f = f(x, y, u)$.

3.4.1 A unified proof of formula (3.1.17)

In this subsection, we assume that $f = f(x, y, u)$ is a heterogenous "ZFK" nonlinearity. Namely, $f = f(x, y, u)$ is a nonlinearity satisfying (3.1.5-3.1.6) and (3.1.8). We will divide the proof of formula (3.1.17) into 3 steps:

Step 1. Under the assumptions (3.1.2), (3.1.3), and (3.1.4) on the domain Ω, the diffusion matrix A, and the advection field q respectively, and having a nonlinearity f satisfying the above assumptions, Theorem 3.1.6 yields that for $c = c^*_{\Omega,A,q,f}(e)$, there exists a solution $u^* = u^*(t, x, y)$ of (4.1.8) such that $u^*_t(t, x, y) > 0$ for all $(t, x, y) \in \mathbb{R} \times \Omega$. In other words, the function ϕ^* defined by

$$\phi^*(s, x, y) = u^*\left(\frac{s - x \cdot e}{c^*(e)}, x, y\right),\ (s, x, y) \in \mathbb{R} \times \overline{\Omega}$$

is increasing in $s \in \mathbb{R}$. Owing to section 3.2, ϕ^* satisfies

$$F[\phi^*] + q \cdot \nabla_{x,y}\phi^* + (q \cdot \tilde{e} - c^*(e))\phi^*_s, + f(x, y, \phi^*) \ = 0 \ \text{ in } \ \mathbb{R} \times \overline{\Omega} \tag{3.4.1}$$

together with boundary and periodicity conditions

$$\begin{cases} \phi^* \text{ is } L\text{-periodic with respect to } x, \\ \nu \cdot A(\nabla_{x,y}\phi^* + \tilde{e}\phi^*_s) = 0 \text{ on } \mathbb{R} \times \overline{\Omega}. \end{cases} \tag{3.4.2}$$

Moreover, (3.4.1) implies that for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$,

$$\begin{aligned} c^*(e) &= \frac{F[\phi^*](s,x,y) + q \cdot \nabla_{x,y}\phi^*(s,x,y) + f(x,y,\phi^*)}{\partial_s \phi^*(s,x,y)} + q(x,y)\cdot \tilde{e} \\ &= R\phi^*(s,x,y), \end{aligned} \tag{3.4.3}$$

and hence

$$c^*(e) \geq \inf_{\varphi \in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} \frac{F[\varphi](s,x,y) + \, q \cdot \nabla_{x,y}\varphi(s,x,y) + f(x,y,\varphi)}{\partial_s \phi(s,x,y)} + q(x,y)\cdot\tilde{e}.$$

In order to prove equality, we argue by contradiction. Assuming that the above inequality is strict, one can find $\delta > 0$ such that

$$c^*(e) - \delta > \inf_{\varphi \in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} \frac{F[\varphi](s,x,y) + \, q \cdot \nabla_{x,y}\varphi(s,x,y) + f(x,y,\varphi)}{\partial_s \varphi(s,x,y)} + q(x,y)\cdot\tilde{e}. \tag{3.4.4}$$

To draw a contradiction, we are going to approach the "ZFK" nonlinearity f by a sequence of "combustion" nonlinearities $(f_\theta)_\theta$ and the minimal speed of propagation by the sequence of speeds $(c_\theta)_\theta$ corresponding to the functions $(f_\theta)_\theta$. The details will appear in the next step.

Step 2. Let χ be a $C^1(\mathbb{R})$ function such that $0 \leq \chi \leq 1$ in $\mathbb{R}$, $\chi(u) = 0$ for all $u \leq 1$, $0 < \chi(u) < 1$ for all $u \in (1,2)$ and $\chi(u) = 1$ for all $u \geq 2$. Assume moreover that χ is non-decreasing in $\mathbb{R}$. For all $\theta \in (0, 1/2)$, let χ_θ be the function defined by

$$\forall\, u \in \mathbb{R}, \quad \chi_\theta(u) = \chi(u/\theta).$$

The function χ_θ is such that $0 \leq \chi_\theta \leq 1$, $0 < \chi_\theta < 1$ in $(-\infty, \theta]$, $0 < \chi_\theta < 1$ in $(\theta, 2\theta)$ and $\chi_\theta = 1$ in $[2\theta, +\infty)$. Furthermore, the functions χ_θ are non-increasing with respect to θ, namely, $\chi_{\theta_1} \geq \chi_{\theta_2}$ if $0 < \theta_1 \leq \theta_2 < 1/2$.

We set

$$f_\theta(x,y,u) = f(x,y,u)\,\chi_\theta(u) \ \text{ for all } (x,y,u) \in \overline{\Omega} \times \mathbb{R}.$$

In other words, we cut off the source term f near $u = 0$.

For each $\theta \in (0, 1/2)$, the function f_θ is a nonlinearity of "combustion" type that satisfies (3.1.5-3.1.6) and (3.1.7) with the ignition temperature θ. Therefore, Theorem 3.1.5 yields that the existence of a classical solution (c_θ, u_θ) of (4.1.8) with the nonlinearity f_θ. Furthermore, the function u_θ is increasing in t and unique up to translation in t and the speed c_θ is unique and positive.

It was proved, through Lemma 6.1 and Lemma 6.2 in Berestycki, Hamel [1], that the speeds c_θ are non-increasing with respect to θ and

$$c_\theta \nearrow c^*(e) \text{ as } \theta \searrow 0.$$

Consider a sequence $\theta_n \searrow 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $c_{\theta_n} \geq c^*(e) - \delta$ for all $n \geq n_0$ (or equivalently $\theta_n \leq \theta_{n_0}$).

In what follows, we fix θ such that $\theta < \theta_{n_0}$. One consequently gets $c_\theta \geq c^*(e) - \delta$. On the other hand, it follows, from the construction of f_θ, that $f \geq f_\theta$ in $\overline{\Omega} \times \mathbb{R}$. Together with (3.4.4), one obtains

$$c_\theta > \inf_{\varphi \in E} \sup_{(s,x,y) \in \mathbb{R}\times\overline{\Omega}} \frac{F[\varphi](s,x,y) + q \cdot \nabla_{x,y}\varphi(s,x,y) + f_\theta(x,y,\varphi)}{\partial_s \varphi(s,x,y)} + q(x,y)\cdot \tilde{e}. \quad (3.4.5)$$

Thus, there exists a function $\psi \in E$ such that

$$c_\theta > \frac{F[\psi](s,x,y) + q \cdot \nabla_{x,y}\psi(s,x,y) + f_\theta(x,y,\psi)}{\partial_s \psi(s,x,y)} + q(x,y)\cdot \tilde{e}. \quad (3.4.6)$$

However, $\psi_s(s,x,y) > 0$ for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$. Thus, the inequality (3.4.6) can be rewritten as

$$L\psi(s,x,y) + f_\theta(x,y,\psi) < 0 \text{ in } \mathbb{R} \times \Omega, \quad (3.4.7)$$

with $\psi \in E$ and L is the operator defined in (3.2.2) for $c = c_\theta$.

For each $\tau \in \mathbb{R}$, we define the function ψ^τ by

$$\psi^\tau(s,x,y) = \psi(s+\tau,x,y) \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Since the coefficients of L are independent of s, the later inequality also holds for all functions ψ^τ with $\tau \in \mathbb{R}$. That is,

$$L\psi^\tau(s,x,y) + f_\theta(x,y,\psi^\tau) < 0 \text{ in } \mathbb{R} \times \overline{\Omega}. \quad (3.4.8)$$

Step 3. For the fixed θ (in step 2), the function f_θ is a "combustion" nonlinearity whose ignition temperature is θ. There corresponds a solution (c_θ, u_θ) of (4.1.8) within the nonlinear source f_θ. We define ϕ_θ by

$$\phi_\theta(s,x,y) = u_\theta\left(\frac{s - x\cdot e}{c_\theta}, x, y\right), \text{ for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Referring to section 3.2, one knows that $\phi_\theta \in E$ and thus it satisfies the following equation

$$L\phi_\theta(s,x,y) + f_\theta(x,y,\phi_\theta) = 0 \ \text{ in } \mathbb{R} \times \overline{\Omega}. \tag{3.4.9}$$

Now, the situation is exactly the same as that in step 2 of the proof of formula (3.1.15) because the nonlinearity f_θ is of "combustion" type. The little difference is that f (in step 2 of the proof of formula (3.1.15)) is replaced here by f_θ, and the function ϕ of equation (3.3.3) is replaced by the function ϕ_θ of (3.4.9). Thus, following the arguments of subsection 3.3.1 and using the same tools of "step 2" as in the proof of formula (3.1.15), one gets that the (3.4.4) is impossible and that completes the proof of formula (3.1.17). □

Remark 3.4.1 *We found that one can use another argument (details are below), different from the sliding method, in order to prove the* $\min - \max$ *formulæ for the speeds of propagation whenever* f *is a homogenous (i.e* $f = f(u)$*) nonlinearity of "combustion" or "ZFK"type and* $\Omega = \mathbb{R}^N$*. Meanwhile, the sliding method, that we used in the proofs of formulæ (3.1.15) and (3.1.17), is a unified argument that works in the general heterogenous periodic framework.*

Another proof of formulæ (3.1.15) and (3.1.17) in a particular framework: Here, we assume that $f = f(u)$, and $\Omega = \mathbb{R}^N$. In fact, following the same procedure of "step 1" in the previous proof, one gets the inequality

$$c^*(e) \geq \inf_{\varphi \in E} \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\varphi(s,x,y).$$

Now, to prove the other sense of inequality, we assume that

$$c^*(e) > \inf_{\varphi \in E} \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\varphi(s,x,y),$$

and we assume that f is of "ZFK" type [2].

Then, as it was explained in "step 2" of the previous proof, one can find $\psi \in E$, $\delta > 0$, $\theta > 0$, and $d > 0$ such that $c^*(e) - \delta < d < c_\theta < c^*(e)$ where

$$\forall (s,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad d > c^*(e) - \delta > R\psi(s,x,y),$$

and $f_\theta(u) = f(u)\,\chi_\theta(u) \leq f(u)$ for all $u \in \mathbb{R}$ is of "combustion" type (c_θ is the speed of propagation, in the direction of $-e$, of pulsating travelling fronts solving (4.1.8) with the nonlinearity f_θ and the domain $\Omega = \mathbb{R}^N$).

2. The case where f is of "combustion" type follows in a similar way.

Hence, for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^N$,

$$d > \frac{F[\psi](s,x,y) + q \cdot \nabla_{x,y}\psi(s,x,y) + f_\theta(\psi)}{\partial_s \psi(s,x,y)} + q(x,y) \cdot \tilde{e}. \tag{3.4.10}$$

Let $\tilde{u}(t,x,y) = \psi(x \cdot e + dt, x, y)$. As it was explained in section 3.2, the function $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \nabla \cdot (A(x,y)\nabla \tilde{u}) - q(x,y) \cdot \nabla \tilde{u} - f_\theta(\tilde{u}) > 0,\ t \in \mathbb{R},\ (x,y) \in \overline{\Omega}, \\ \nu \cdot A\ \nabla \tilde{u}(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega, \\ \forall\, k \in \prod_{i=1}^{d} L_i\mathbb{Z},\ \forall\, (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad \tilde{u}(t + \frac{k \cdot e}{d}, x, y) = \tilde{u}(t, x+k, y), \\ 0 \le \tilde{u} \le 1. \end{cases} \tag{3.4.11}$$

Let $0 \le u_0(x,y) \le 1$ be a function in $C(\mathbb{R}^N)$ such that $u_0(x,y) \to 0$ as $x \cdot e \to -\infty$, and $u_0(x,y) \to 1$ as $x \cdot e \to +\infty$, uniformly in y and all directions of $\mathbb{R}^d$ which are orthogonal to e. Let u be a pulsating front propagating in the direction of $-e$ with the speed c_θ and solving the initial data problem

$$\begin{cases} u_t = \nabla \cdot (A(x,y)\nabla u) + q(x,y) \cdot \nabla u + f_\theta(u),\ t > 0,\ (x,y) \in \overline{\Omega}, \\ u(0,x,y) = u_0(x,y), \\ \nu \cdot A\ \nabla u(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega. \end{cases} \tag{3.4.12}$$

Having $f_\theta(u)$ as a "combustion" nonlinearity, it follows from J. Xin [19] (Theorem 3.5) and Weinberger [18], that

$$\begin{aligned} \forall r > 0, \quad & \lim_{t \to +\infty} \sup_{|x| \le r} u(t, x - cte, y) = 0 \text{ uniformly in } y, \text{ for every } c > c_\theta, \\ \text{and } & \lim_{t \to +\infty} \inf_{|x| \le r} u(t, x - cte, y) = 1 \text{ uniformly in } y, \text{ for every } c < c_\theta. \end{aligned} \tag{3.4.13}$$

This means that the speed of propagation c_θ corresponding to (4.1.8) is equal to the spreading speed in the direction of $-e$ when the nonlinearity is of "combustion" type and the initial data u_0 satisfies the above conditions.

For all $(t,x,y) \in [0,+\infty) \times \overline{\Omega}$, let $w(t,x,y) = \tilde{u}(t,x,y) - u(t,x,y)$. It follows, from

(3.4.16) and (3.4.17), that

$$\begin{cases} w_t - \nabla\cdot(A(x,y)\nabla w) - q(x,y)\cdot\nabla w + bw > 0,\ t>0,\ (x,y)\in\overline{\Omega}, \\ \forall (x,y)\in\overline{\Omega},\ w(0,x,y)\geq 0, \\ \nu\cdot A\,\nabla w(t,x,y)=0,\ t\in\mathbb{R},\ (x,y)\in\partial\Omega, \end{cases} \tag{3.4.14}$$

for some $b=b(t,x,y)\in C(\mathbb{R}\times\overline{\Omega})$. The parabolic maximum principle implies that $w\geq 0$ in $[0,+\infty)\times\overline{\Omega}$. In other words,

$$\forall (t,x,y)\in[0,+\infty)\times\overline{\Omega},\quad u(t,x,y)\leq \tilde{u}(t,x,y).$$

However, for all $c>d$,

$$\lim_{t\to+\infty}\tilde{u}(t,x-cte,y)=\lim_{t\to+\infty}\psi(x\cdot e+(d-c)t,x-cte,y)=0$$

locally in x and uniformly in y (since $\psi\in E$). Consequently,

$$\forall c>d,\ \forall r>0,\quad \lim_{t\to+\infty}\sup_{|x|\leq r}u(t,x-cte,y)=0 \text{ uniformly in } y.$$

Referring to (3.4.18), one concludes that $d\geq c_\theta$ which is impossible ($d<c_\theta$). Therefore, our assumption that $c^*(e)>\inf_{\varphi\in E}\sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}}R\varphi(s,x,y)$ is false and that completes the proof of formula (3.1.17) in the case where $f=f(u)$ and $\Omega=\mathbb{R}^N$. □

3.4.2 Another proof of formula (3.1.17) under an additional assumption on f

In the subsection 3.4.1, we proved formula (3.1.17) in the case where the nonlinearity f is a general $L-$periodic with respect to x “ZFK” nonlinearity. The proof of this formula becomes simpler if we add an assumption of non-degeneracy on the nonlinearity f at $u=0$. Precisely, we assume that the nonlinearity f satisfies (3.1.5-3.1.6), (3.1.8) together with the additional assumption

$$\liminf_{u\to 0^+}\frac{f(x,y,u)}{u}>0. \tag{3.4.15}$$

First, following the same procedure of “step 1” in subsection 3.4.1, one gets the inequality

$$c^*(e) \geq \inf_{\varphi \in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y).$$

Now, to prove the other sense of inequality, we assume that

$$c^*(e) > \inf_{\varphi \in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y).$$

Then, one can find $0 < d < c^*(e)$ and $\psi \in E$ such that

$$\forall (s,x,y) \in \mathbb{R}\times\overline{\Omega}, \quad d > R\psi(s,x,y).$$

For each $(t,x,y) \in \mathbb{R}\times\overline{\Omega}$, let $\tilde{u}(t,x,y) = \psi(x\cdot e + dt, x, y)$. It follows from section 3.2 that the function $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \nabla\cdot(A(x,y)\nabla\tilde{u}) - q(x,y)\cdot\nabla\tilde{u} - f(x,y,\tilde{u}) > 0,\ t\in\mathbb{R},\ (x,y)\in\overline{\Omega}, \\ \nu\cdot A\,\nabla\tilde{u}(t,x,y) = 0,\ t\in\mathbb{R},\ (x,y)\in\partial\Omega, \\ \forall\, k \in \prod_{i=1}^{d} L_i\mathbb{Z},\ \forall\,(t,x,y)\in\mathbb{R}\times\overline{\Omega},\quad \tilde{u}(t+\dfrac{k\cdot e}{d},x,y) = \tilde{u}(t,x+k,y), \\ 0 \leq \tilde{u} \leq 1. \end{cases} \tag{3.4.16}$$

Let $u_0(x,y)$ be a nonnegative function in $C(\overline{\Omega})$ such that $u_0(x,y) = 0$ for all (x,y) in $\overline{\Omega}$ with $x\cdot e \leq 0$, $\inf_{(x,y)\in\Omega,\ x\cdot e > L} u_0(x,y) > 0$ (for some $L > 0$) and such that

$$\forall (x,y)\in\overline{\Omega}, \quad u_0(x,y) \leq \tilde{u}(0,x,y).$$

Let u be a classical solution of the problem

$$\begin{cases} u_t = \nabla\cdot(A(x,y)\nabla u) + q(x,y)\cdot\nabla u + f(x,y,u),\ t>0,\ (x,y)\in\overline{\Omega}, \\ u(0,x,y) = u_0(x,y), \\ \nu\cdot A\,\nabla u(t,x,y) = 0,\ t\in\mathbb{R},\ (x,y)\in\partial\Omega. \end{cases} \tag{3.4.17}$$

Under the conditions (3.1.5-3.1.6), (3.1.8) and (3.4.15) on the nonlinearity f, the results of Weinberger [18] imply that

$$\begin{aligned} &\forall r > 0,\ \lim_{t\to+\infty} \sup_{|x|\leq r,(x-cte,y)\in\overline{\Omega}} u(t,x-cte,y) = 0 \text{ for every } c > c^*(e), \\ &\quad\text{and} \lim_{t\to+\infty} \inf_{|x|\leq r,(x-cte,y)\in\overline{\Omega}} u(t,x-cte,y) = 1 \text{ for every } c < c^*(e). \end{aligned} \tag{3.4.18}$$

This means that the minimal speed of propagation $c^*(e)$ corresponding to (4.1.8) is equal to the spreading speed when the "ZFK" nonlinearity f satisfies (3.4.15) and u_0 satisfies the above conditions.

For all $(t,x,y)\in\mathbb{R}\times\overline{\Omega}$, let $w(t,x,y)=\tilde{u}(t,x,y)-u(t,x,y)$. It follows, from (3.4.16) and (3.4.17), that

$$\begin{cases} w_t-\nabla\cdot(A(x,y)\nabla w)-q(x,y)\cdot\nabla w+\ b(t,x,y)w>0,\ t>0,(x,y)\ \in\ \overline{\Omega},\\ \forall(x,y)\in\overline{\Omega},\ w(0,x,y)\geq 0,\\ \nu\cdot A\ \nabla w(t,x,y)=0,\ t\geq 0,\ (x,y)\ \in\ \partial\Omega, \end{cases}\tag{3.4.19}$$

for some $b=b(t,x,y)\in C([0,+\infty)\times\overline{\Omega})$. The parabolic maximum principle implies that $w\geq 0$ in $[0,+\infty)\times\overline{\Omega}$. In other words,

$$\forall(t,x,y)\in[0,+\infty)\times\overline{\Omega},\quad u(t,x,y)\leq\tilde{u}(t,x,y).$$

However, for all $c>d$,

$$\lim_{t\to+\infty}\tilde{u}(t,x-cte,y)=\lim_{t\to+\infty}\psi(x\cdot e+(d-c)t,x-cte,y)=0$$

locally in x and uniformly in y (since $\psi\in E$). Consequently,

$$\forall c>d,\ \forall r>0,\quad \lim_{t\to+\infty}\ \sup_{|x|\leq r,(x-cte,y)\in\overline{\Omega}} u(t,x-cte,y)=0.$$

Referring to (3.4.18), one concludes that $d\geq c^*(e)$ which is impossible ($d<c^*(e)$). Therefore, our assumption that $c^*(e)>\inf_{\varphi\in E}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y)$ is false and that completes the proof of formula (3.1.17) in the case where $\liminf_{u\to 0^+}\dfrac{f(x,y,u)}{u}>0$ for all $(x,y)\in\overline{\Omega}$. $\square$

3.5 Another proof of formulæ (3.2.7) and (3.2.9) using sub and super solutions arguments in regularized elliptic equations

In this section, the nonlinear source f can be of "combustion" or "ZFK" type. **Moreover, we assume that the restriction of the nonlinearity f in (4.1.8) is** $C^{1,\delta}(\overline{\Omega}\times[0,1])$. In fact, this assumption insures that any classical solution $u=u(t,x,y)$

of a reaction-advection-diffusion equation, having $f = f(x,y,u)$ as a nonlinearity, is of class $C^2(\mathbb{R}\times\overline{\Omega})$. In Remark 3.2.2, the proof of the $\min-\max$ formulæ (3.2.7) and (3.2.9) was direct owing to the $\min-\max$ formulæ (3.1.15-3.1.17) and to the estimate (3.2.6). In what follows, we are going to give another proof of these formulæ (over the subset E' of E) without using formulæ (3.1.15) and (3.1.17).

As it was mentioned in Remark 3.2.2, there exists a constant $M>0$ such that

$$\forall (s,x,y)\in\mathbb{R}\times\overline{\Omega},\ \ |\partial_{ss}\phi(s,x,y)|\le M\ \partial_s\phi(s,x,y), \tag{3.5.1}$$

where $\phi(s,x,y)=u\left(\dfrac{s-x\cdot e}{c(e)},x,y\right)$ and $(c(e),u)$ is the unique pulsating travelling front solving (4.1.8) with a "combustion" nonlinearity. Similarly, there exists a constant $M^*>0$ such that

$$\forall (s,x,y)\in\mathbb{R}\times\overline{\Omega},\ \ |\partial_{ss}\phi^*(s,x,y)|\le M^*\ \partial_s\phi^*(s,x,y), \tag{3.5.2}$$

where $\phi^*(s,x,y)=u^*\left(\dfrac{s-x\cdot e}{c^*(e)},x,y\right)$ and $(c^*(e),u^*)$ is the pulsating travelling front solving (4.1.8) with a "ZFK" nonlinearity. Consequently, $\phi\in E'$ and $\phi^*\in E'$ and they satisfy (see section 3.2)

$$\forall (s,x,y)\in\mathbb{R}\times\overline{\Omega},\ c(e)=R\phi(s,x,y)\text{ and }c^*(e)=R\phi^*(s,x,y).$$

Thus,

$$c(e)\ge\inf_{\varphi\in E'}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}}R\varphi(s,x,y)$$

when f is of the "combustion" type, and

$$c^*(e)\ge\inf_{\varphi\in E'}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}}R\varphi(s,x,y)$$

when f is of the "ZFK" type.

To complete the proofs of formulæ (3.2.7) and (3.2.9), we assume that the above two inequalities are strict and we search a contradiction. However, in what follows, we will consider $f=f(x,y,u)$ as a "ZFK" nonlinearity whose restriction is of class $C^{1,\delta}(\overline{\Omega}\times[0,1])$. In fact, the same ideas can be imitated in the "combustion" case after replacing $c^*(e)$ by $c(e)$, u^* by u and ϕ^* by ϕ.

Now, after assuming that $c^*(e)>\inf\limits_{\varphi\in E'}\ \sup\limits_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}}R\varphi(s,x,y)$, there exists $\psi\in E'$

such that

$$\forall\,(s,x,y)\in\mathbb{R}\times\overline{\Omega},\quad c^*(e)>R\psi(s,x,y).$$

Then, one can find $0<d'<c^*(e)$ such that

$$\forall\,(s,x,y)\in\mathbb{R}\times\overline{\Omega},\quad c^*(e)>d'>R\psi(s,x,y).$$

However, $\dfrac{\psi_{ss}}{\psi_s}$ is bounded over $\mathbb{R}\times\overline{\Omega}$ ($\psi\in E'$). Hence, there exists $\varepsilon_0>0$ (small enough) and $0<d'<d<c^*(e)$ such that

$$\forall\,0<\varepsilon\le\varepsilon_0,\;\;\forall(s,x,y)\in\mathbb{R}\times\overline{\Omega},\;d\ge\frac{\varepsilon\psi_{ss}(s,x,y)}{\psi_s(s,x,y)}+R\psi(s,x,y).\tag{3.5.3}$$

In other words,

$$\forall\,0<\varepsilon\le\varepsilon_0,\;\forall\,(s,x,y)\in\mathbb{R}\times\Omega,\;\;\varepsilon\psi_{ss}(s,x,y)+L_d\,\psi(s,x,y)+f(x,y,\psi)\le 0,\tag{3.5.4}$$

where

$$L_d=\nabla_{x,y}\cdot(A\nabla_{x,y})+(\tilde{e}A\tilde{e})\partial_{ss}+\nabla_{x,y}\cdot(A\tilde{e}\partial_s)+\partial_s(\tilde{e}A\nabla_{x,y})+q\cdot\nabla_{x,y}+(q\cdot\tilde{e}-d)\partial_s.$$

We notice that L_d is a degenerated elliptic operator while

$$\begin{aligned}\varepsilon\partial_{ss}+L_d\;&=\;(\tilde{e}A\tilde{e}+\varepsilon)\partial_{ss}+\nabla_{x,y}\cdot(A\nabla_{x,y})+\nabla_{x,y}\cdot(A\tilde{e}\,\partial_s)\\&\quad+\partial_s(\tilde{e}A\nabla_{x,y})+\,q\cdot\nabla_{x,y}\,+\,(q\cdot\tilde{e}\,-\,d)\partial_s\end{aligned}$$

is a uniformly elliptic operator. Thus, $\varepsilon\psi_{ss}$ plays the role of a regularizing term in (3.5.4).

For each $0<\varepsilon\le\varepsilon_0$ (small enough), we consider the uniformly elliptic problem (with generalized Neumann boundary conditions)

$$(P_\varepsilon)\begin{cases}\varepsilon\partial_{ss}w^\varepsilon(s,x,y)+L_d\,w^\varepsilon(s,x,y)+f(x,y,w^\varepsilon)=0\text{ in }\mathbb{R}\times\Omega,\\ \nu\cdot(\tilde{e}w^\varepsilon_s+\nabla_{x,y}w^\varepsilon)=0\text{ on }\mathbb{R}\times\partial\Omega,\\ w^\varepsilon\text{ is }L\text{-periodic with respect to }x,\\ w^\varepsilon(-\infty,x,y)=0\text{ and }w^\varepsilon(+\infty,x,y)=1\text{ uniformly in }(x,y)\in\overline{\Omega},\\ 0\le w^\varepsilon\le\psi<1\text{ in }\mathbb{R}\times\overline{\Omega},\\ w^\varepsilon_s>0\text{ in }\mathbb{R}\times\overline{\Omega},\end{cases}\tag{3.5.5}$$

with the normalization condition

$$\max_{(x,y)\in\overline{\Omega}} w^{\varepsilon}(0,x,y) = \frac{1}{2}. \tag{3.5.6}$$

In the "combustion" case, we consider the same problem (3.5.5) but with the normalization condition

$$\max_{(x,y)\in\overline{\Omega}} w^{\varepsilon}(0,x,y) = \theta, \tag{3.5.7}$$

where θ is the ignition temperature of f (see (3.1.7)).

We have the following

Lemma 3.5.1 *For each $\varepsilon > 0$, (3.5.5) admits a classical solution $w^{\varepsilon} = w^{\varepsilon}(s,x,y)$ satisfying (3.5.6) in the "ZFK" case or (3.5.7) in the "combustion" case. Moreover, taking in both cases, $v^{\varepsilon}(t,x,y) = w^{\varepsilon}(x\cdot e + dt, x, y)$, then for each compact subset $\mathcal{K}$ of $\overline{\Omega}$, there exists a constant $C(\mathcal{K})$, only depending on $\mathcal{K}$, such that*

$$\int_{\mathbb{R}\times\mathcal{K}} \left[(v^{\varepsilon}_t)^2 + |\nabla_{x,y} v^{\varepsilon}|^2\right] dt\,dx\,dy \le C(\mathcal{K})\left(\frac{1+\|q\|^2_{\infty}}{2\alpha_1} + 2\max_{(x,y)\in\overline{\Omega}} F(x,y,1)\right), \tag{3.5.8}$$

where $\alpha_1 > 0$ is given in (3.1.3) and $F(x,y,t) = \displaystyle\int_0^t f(x,y,\tau)\,d\tau$.

In fact, the "a priori" estimate (3.5.8) was given in Lemma 5.11 of Berestycki, Hamel [1]. The proof of Lemma 3.5.1 will be postponed to the end of this section.

Going back to the proof of formula (3.2.9), we call (for each $0 < \varepsilon \le \varepsilon_0$)

$$v^{\varepsilon}(t,x,y) = w^{\varepsilon}(x\cdot e + dt, x, y), \quad \text{for all } (t,x,y) \in \mathbb{R}\times\overline{\Omega},$$

where $w^{\varepsilon} = w^{\varepsilon}(s,x,y)$ is a classical solution of (3.5.5) whose existence follows from Lemma 3.5.1. It follows from (3.5.5) that for each ε, v^{ε} is a classical solution of

$$\begin{cases} v^{\varepsilon}_t + \nabla\cdot(A\,\nabla_{x,y} v^{\varepsilon}) + \dfrac{\varepsilon}{d^2} v^{\varepsilon}_{tt} + q\cdot\nabla_{x,y} v^{\varepsilon} + f(x,y,v^{\varepsilon}) = 0 \text{ in } \mathbb{R}\times\overline{\Omega}, \\ \nu\cdot A\nabla_{x,y} v^{\varepsilon} = 0 \text{ on } \mathbb{R}\times\partial\Omega, \\ \forall\, k \in \displaystyle\prod_{i=1}^{i=d} L_i\mathbb{Z},\ \forall\,(t,x,y) \in \mathbb{R}\times\overline{\Omega}, v^{\varepsilon}(t+\dfrac{k\cdot e}{d}, x, y) = v^{\varepsilon}(t, x+k, y), \\ \displaystyle\max_{x\cdot e = -dt,\,(t,x,y)\in\mathbb{R}\times\overline{\Omega}} v^{\varepsilon}(t,x,y) = \dfrac{1}{2}. \end{cases} \tag{3.5.9}$$

Moreover, each v^{ε} is increasing in t since $v^{\varepsilon}_t = dw^{\varepsilon}_s$ in $\mathbb{R}\times\overline{\Omega}$ and w^{ε} is increasing in s. Furthermore, $v^{\varepsilon}(t,x,y) \to 0$ (resp. $v^{\varepsilon}(t,x,y) \to 1$) as $t \to -\infty$ (resp. $t \to +\infty$) in

$C^2_{loc}(\overline{\Omega})$.

Let $\{\varepsilon_n\}_n \in (0, \varepsilon_0]$ be a sequence such that $\varepsilon_n \to 0^+$ as $n \to +\infty$. It follows, from (3.5.8), that $\{v^{\varepsilon_n}\}_n$ is bounded in $H^1(\mathbb{R} \times \mathcal{K})$ for every compact $\mathcal{K} \subset \overline{\Omega}$. Thus, there exists a function $v \in H^1_{loc}(\mathbb{R} \times \overline{\Omega})$ such that, up to extraction of some subsequence, the functions v^{ε_n} satisfy: $v^{\varepsilon_n} \to v$ strongly in $L^2_{loc}(\mathbb{R} \times \overline{\Omega})$, $v^{\varepsilon_n} \rightharpoonup v$ weakly in $H^1_{loc}(\mathbb{R} \times \overline{\Omega})$ and $v^{\varepsilon_n} \to v$ a.e in $\mathbb{R} \times \overline{\Omega}$ as $n \to +\infty$. From parabolic regularity, the function v is then a classical solution of

$$\begin{cases} v_t + \nabla \cdot (A \nabla_{x,y} v) + q \cdot \nabla_{x,y} v \, + f(x,y,v) = 0 \text{ in } \mathbb{R} \times \overline{\Omega}, \\ \nu \cdot A \nabla_{x,y} v \; = \; 0 \quad \text{on } \mathbb{R} \times \partial\Omega, \\ \forall \, k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}, \; \forall (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \; v(t + \dfrac{k \cdot e}{d}, x, y) \; = v(t, x+k, y), \\ 0 \leq v \leq 1 \text{ and } v_t \, \geq \, 0 \text{ in } \mathbb{R} \times \overline{\Omega}. \end{cases} \tag{3.5.10}$$

Furthermore, from the normalization of v^{ε} on the set $\{x \cdot e \, = \, -dt\}$ and from the monotonicity of the functions v^{ε} in t, it follows that

$$v(t,x,y) \, \leq \, \frac{1}{2} \text{ for all } (t,x,y) \text{ such that } x \cdot e \, \leq \, -dt. \tag{3.5.11}$$

On the other hand, (3.5.9) is an elliptic regularization of a parabolic equation. From Theorem A.1 in [1] (it is easy to check that the assumptions are satisfied, especially the functions v^{ε} are $C^3(\,\mathbb{R} \, \times \, \overline{\Omega})$ from the regularity assumptions and from standard elliptic estimates), the following gradient estimates hold :

$$||\nabla_{x,y} v^{\varepsilon}||_{L^{\infty}\,(\mathbb{R} \times \overline{\Omega})} \, \leq \, C, \tag{3.5.12}$$

where C is independent of ε.

Since $\max_{x.e \, = \, -dt, \, (t,x,y) \, \in \mathbb{R} \times \overline{\Omega}} v^{\varepsilon}(t,x,y) = 1/2$, and $v^{\varepsilon}(t{+}k{\cdot}\, e/c, x, y) = v^{\varepsilon}(t, x+k, y)$ in $\mathbb{R} \, \times \, \overline{\Omega}$ for all $k \, \in \, \prod_{i=1}^{d} L_i\mathbb{Z}$, (hence it suffices to consider v^{ε} on a set which is bounded in the (t,x,y) variables), it follows that there exists a bounded sequence $(t_n, x_n, y_n) \in \mathbb{R} \times \overline{C}$ such that $x_n \cdot e \, = \, -d\, t_n$ and $v^{\varepsilon_n}(t_n, x_n, y_n) \, = \, 1/2$. Therefore, the sequence (t_n, x_n, y_n) converges, up to extraction of some subsequence, to a point $(\bar{t}, \bar{x}, \bar{y}) \in \mathbb{R} \times \overline{C}$ such that $\bar{x} \cdot e \, = \, -d\,\bar{t}$. We claim that $v(\bar{t}, \bar{x}, \bar{y}) \geq 1/2$. Thus, we fix $\eta > 0$, and we take $0 < r \leq \eta/C$. From the uniform gradient estimates (3.5.12), we have $v^{\varepsilon_n}(t_n, x, y) \geq 1/2 - \eta$, for all n and for all $(x,y) \, \in \, B_r(x_n, y_n) \cap \overline{\Omega}$, where $B_r(x_n, y_n)$ denotes the euclidian closed ball in $\mathbb{R}^N$ of radius r and center (x_n, y_n). Consequently, as v^{ε} is increasing in t, it follows that for n large enough,

$$v^{\varepsilon_n}(t,x,y) \geq 1/2-\eta \ \text{ for all } \ t \geq t_n \text{ and for all } \ (x,y) \in B_{r/2}(\bar{x},\bar{y}).$$

Since the functions v^{ε_n} converge almost everywhere to the continuous function v, then

$$v(t,x,y) \geq 1/2-\eta \ \text{ for all } \ t \geq \bar{t} \ \text{ and } \ \text{ for all } \ (x,y) \in B_{r/2}(\bar{x},\bar{y}).$$

However, $\eta > 0$ was arbitrary. It follows that $v(\bar{t},\bar{x},\bar{y}) \geq 1/2$. From (3.5.11) and the (t,x) periodicity of the function v, one concludes that

$$\max_{x.e=-d\,t,\,(x,y)\in\mathbb{R}\times\overline{\Omega}} v(t,x,y) = 1/2. \tag{3.5.13}$$

Lastly, from standard parabolic estimates (the coefficients of the parabolic equation (3.5.10) are independent of t) and from the monotonicity of v with respect to t, one gets that $v(t,x,y) \to v_{\pm(x,y)}$ in $C^2_{loc}(\overline{\Omega})$ as $t \to \pm\infty$, and the functions $v_{\pm}$ satisfy

$$\begin{cases} \nabla\cdot(A\nabla v_{\pm}) + q.\nabla\, v_{\pm} + f(x,y,v_{\pm}) = 0 \quad \text{in} \quad \overline{\Omega} \\ \nu\cdot A\nabla v_{\pm} = 0 \text{ on } \partial\Omega \\ v_{\pm} \text{ is } L- \text{ periodic with respect to } x, \end{cases} \tag{3.5.14}$$

and $0 \leq v_- \leq v_+ \leq 1$. Integrating the first equation of (3.5.14) by parts over C and owing to (3.1.4), one then obtains

$$\int_C f(x,y,v_{\pm})\,dx\,dy = 0.$$

Since f is nonnegative and continuous over $\overline{\Omega}\times[0,1]$, it follows that $f(x,y,v_{\pm}) = 0$ for all $(x,y) \in \overline{\Omega}$. Now, we multiply (3.5.14) by $v_{\pm}$ and integrate by parts over C. It follows that

$$\int_C \nabla v_{\pm}\cdot A\nabla v_{\pm} = 0.$$

Together with (3.1.3), one gets that $v_{\pm}$ are constants. Moreover, it follows from (3.5.13) and from the monotonicity of v that $0 \leq v_- \leq 1/2 \leq v_+ \leq 1$. The nonlinearity $f = f(x,y,u)$ is a "ZFK" nonlinearity (satisfying (3.1.5) and (3.1.8)). Consequently, $v_- = 0$ and $v_+ = 1$.

Furthermore, the (t,x) periodicity of the classical solution v of (3.5.10) together with the above notes imply that

$$\lim_{x\cdot e\to+\infty} v(t,x,y) = 1 \text{ and } \lim_{x\cdot e\to-\infty} v(t,x,y) = 0$$

uniformly in y and all directions of $\mathbb{R}^d$ which are orthogonal to e (in fact, for all y, a sequence $\{(x_n,y)\}_n \in \overline{\Omega}$ such that $x_n\cdot e \to \pm\infty$ as $n\to+\infty$ can be written as $(x_n = \tilde{x}_n + k_n, y)$ for all n, where $\tilde{x}_n \in \bar{C}$ and $k_n \in L_1\mathbb{Z}\times\cdots L_d\mathbb{Z}$. Thus, $k_n\cdot e\to\pm\infty$ as $x_n\cdot e\to\pm\infty$ because the sequence $(\tilde{x}_n\cdot e)_n$ will be bounded. Owing to the above notes, one then has $v(t,x_n,y) = v(t+\dfrac{k_n\cdot e}{d},\tilde{x}_n,y)\to 0$ (resp. 1) as $x_n\cdot e\to-\infty$ (resp. $x_n\cdot e\to+\infty$) locally in t and uniformly in y and all directions of $\mathbb{R}^d$ which are orthogonal to e).

Eventually, the function $v=v(t,x,y)$ is a pulsating travelling front solving (4.1.8) and propagating with a speed $d<c^*(e)$ which contradicts the minimality of $c^*(e)$ in the "ZFK" case. Therefore, our assumption that $c^*(e) > \inf_{\varphi\in E'} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y)$ is false and that completes the proof of formula (3.2.9). □

Remark 3.5.2 (About the proof of (3.2.7) in the "combustion" case) *If $f = f(x,y,u)$ is a "combustion" nonlinearity satisfying (3.1.5-3.1.6) and (3.1.7) with an ignition temperature θ, and whose restriction is $C^1(\overline{\Omega}\times[0,1])$, we imitate the above proof of formula (3.2.9) in order to prove formula (3.2.7). However, there will be a difference in the normalization conditions that we assume on the functions w^ε and v^ε to avoid trivial solutions. In details, we replace the condition $\max_{(x,y)\in\overline{\Omega}} w^\varepsilon(0,x,y) = 1/2$ in (3.5.5) by the condition $\max_{(x,y)\in\overline{\Omega}} w^\varepsilon(0,x,y)=\theta$ and this leads to the condition*

$$\max_{x\cdot e\,=\,-dt,\,(t,x,y)\,\in\,\mathbb{R}\times\,\overline{\Omega}} v^\varepsilon(t,x,y) \;=\; \theta.$$

Then, using the same gradient estimates (3.5.12) together parabolic regularity results and uniform parabolic estimates, the functions $v^{\varepsilon_n}\to v$ (weakly in $H^1_{loc}(\mathbb{R}\times\overline{\Omega})$ and strongly in $L^2_{loc}(\mathbb{R}\times\overline{\Omega})$) as $\varepsilon_n\to 0^+$, where v will be a classical solution of the same problem (3.5.10) but with the normalization condition

$$\max_{x\cdot e\,=\,-dt,\,(t,x,y)\,\in\,\mathbb{R}\times\,\overline{\Omega}} v(t,x,y) \;=\; \theta.$$

Thus, $v_\pm$ will be constants and they will satisfy

$$\forall\,(x,y)\in\overline{\Omega},\;\; f(x,y,v_\pm)=0,$$

with $0 \leq v_- \leq v_+ \leq 1$*. Moreover, the function* $v(t,x,y)$ *will satisfy (as it was done in section 5 of [1], equation (5.47))*

$$\forall (t,x,y) \in \mathbb{R} \times \overline{\Omega},\ x \cdot e + dt \leq 0 \Rightarrow v(t,x,y) \leq e^{\lambda(x\cdot e+dt)} \varphi(x,y), \tag{3.5.15}$$

for some positive $C^2(\overline{\Omega})$ *function* φ *which is* $L-$*periodic with respect to* x *such that* $\min_{(x,y)\in\overline{\Omega}} \varphi(x,y) = \min_{(x,y)\in\overline{C}} \varphi(x,y) = \theta$*. Consequently,* $v(t,x,y) \to 0$ *as* $t \to -\infty$ *or as* $x \cdot e \to -\infty$ *(locally in* t*) uniformly in* y *and in the directions of* $\mathbb{R}^d$ *orthogonal to* e*. Thus,* $v_- = 0$*.*

If $v_+ \leq \theta$*, then the maximum principle and the normalization condition* $\max_{x\cdot e = -dt,\,(t,x,y)\in\mathbb{R}\times\overline{\Omega}} v(t,x,y) = \theta$ *will yield that* $v(t,x,y) = \theta$ *in* $\mathbb{R}\times\overline{\Omega}$*. However, this will contradict the fact that* $v_- = 0$*. Thus, one should have* $v_+ > \theta$ *and* $f(x,y,v_+) = 0$*. From (3.1.7), one should have* $v_+ = 1$*.*

Eventually, the function $v = v(t,x,y)$ *will be a pulsating travelling front solving (4.1.8) taken with the "combustion" nonlinearity* f *whose speed equals* $d < c(e)$*. However, this will contradict with the uniqueness of the pulsating front* $(c(e),u)$ *that follows from Theorem 3.1.5.* □

The following result will be needed in the proof of Lemma 3.5.1:

Lemma 3.5.3 *Let* $c \in \mathbb{R}$*,* a *and* ε *be two positive real numbers, and let*

$$\Sigma_a = (-a,a) \times \Omega \quad \text{and} \quad \widetilde{\Sigma_a} = \overline{\Sigma_a} \setminus (\{\pm a\} \times \partial\Omega).$$

Assume that $f = f(x,y,u)$ *is a nonnegative Lipschitz-continuous nonlinearity defined on* $\mathbb{R}\times\overline{\Omega}$*, which is* $L-$*periodic with respect* x *and such that* $f = 0$ *in* $\overline{\Omega} \times (-\infty,0] \cup [1,+\infty)$*. Assume, furthermore, that* A *and* $q = q(x,y)$ *satisfy (3.1.3) and (3.1.4) respectively. Let* φ *be a solution in* $C(\overline{\Sigma_a}) \cap C^2(\widetilde{\Sigma_a})$ *of*

$$\begin{cases} L^{\varepsilon,c}\varphi + f(x,y,\varphi) = 0 \ \text{ in } \Sigma_a, \\ \nu \cdot A(\tilde{e}\varphi_s + \nabla_{x,y}\varphi) = 0 \quad \text{on } (-a,a) \times \partial\Omega, \\ \varphi \text{ is } L-\text{periodic with respect to } x, \\ \forall\,(x,y) \in \overline{\Omega},\ \varphi(-a,x,y) = 0 \ \text{ and } \ \varphi(a,x,y) = \tilde{\varphi}(a,x,y), \\ \varphi \leq \tilde{\varphi} \text{ in } \overline{\Sigma_a}, \end{cases} \tag{3.5.16}$$

where

$$\begin{aligned} L^{\varepsilon,c} &= (\tilde{e}A\tilde{e} + \varepsilon)\partial_{ss} + \nabla_{x,y}\cdot(A\nabla_{x,y}) + \nabla_{x,y}\cdot(A\tilde{e}\,\partial_s) \\ &\quad + \partial_s(\tilde{e}A\nabla_{x,y}) + q\cdot\nabla_{x,y} + (q\cdot\tilde{e} - c)\partial_s \end{aligned}$$

is a regularized elliptic operator, and $\tilde{\varphi} = \tilde{\varphi}(s,x,y)$ *is defined over* $\overline{\Sigma_a}$ *such that*

$$\begin{cases} L^{\varepsilon,c}\tilde{\varphi} \, + \, f(x,y,\tilde{\varphi}) \, \leq \, 0 \quad \text{in } \Sigma_a, \\ \nu \cdot A(\tilde{e}\tilde{\varphi}_s \, + \, \nabla_{x,y}\tilde{\varphi}) \, = \, 0 \quad \text{on } (-a,a) \, \times \, \partial\Omega, \\ \tilde{\varphi} \text{ is } L\text{-periodic with respect to } \; x, \\ \tilde{\varphi} > 0 \text{ in } \overline{\Sigma_a} \text{ and } \tilde{\varphi} \text{ is nondecreasing with respect to } s. \end{cases} \tag{3.5.17}$$

Then, the function $\varphi \, = \, \varphi(s,x,y)$ *is increasing in* s *and it is the unique solution of (3.5.16) in* $C(\overline{\Sigma_a}) \cap C^2(\widetilde{\Sigma_a})$.

Proof. We use, in this proof, the sliding method of Berestycki and Nirenberg [4].

First, let us mention that since $f = 0$ in $\overline{\Omega} \times (-\infty, 0] \cup [1, +\infty)$, then the elliptic maximum principle and the Hopf lemma yield that $0 < \varphi < 1$ in $(-a,a) \times \overline{\Omega}$. Furthermore, the maximum principle and the Hopf lemma, applied on the function $\varphi - \tilde{\varphi}$, yield that

$$0 < \varphi(s,x,y) < \tilde{\varphi}(s,x,y) \text{ for all } (s,x,y) \in (-a,a) \times \overline{\Omega}.$$

For any $\lambda \in (0,2a)$, let φ^{λ} be the function defined by

$$\forall \, (s,x,y) \in \overline{\Sigma_a^{\lambda}}, \;\; \varphi^{\lambda}(s,x,y) \; = \; \varphi(s + 2a - \lambda, x, y),$$

with $\Sigma_a^{\lambda} := (-a, -a + \lambda) \times \Omega$.

In order to prove that φ is increasing with respect to the variable s in $\overline{\Sigma_a}$, it suffices to prove that

$$\varphi < \varphi^{\lambda} \text{ in } \overline{\Sigma_a^{\lambda}} \text{ for all } \lambda \in (0,2a). \tag{3.5.18}$$

Owing to the continuity and the $L-$periodicity, with respect to x, of the function φ and since $\varphi(-a,.,.) = 0$ and $\varphi(a,.,.) = 1$, it follows that (3.5.18) is true for small λ.

Let us now increase λ and set

$$\lambda^* = \sup\{\lambda \in (0,2a); \; \varphi < \varphi^{\mu} \text{ in } \overline{\Sigma_a^{\lambda}} \text{ for all } \mu \in (0,\lambda)\} > 0.$$

To complete the proof, we are going to prove that $\lambda^* = 2a$. Thus, we assume, to the contrary, that $\lambda^* < 2a$. By continuity, one has $\varphi \leq \varphi^{\lambda^*}$ in $\overline{\Sigma_a^{\lambda^*}}$. On the other hand, there exist two sequences $\lambda_n \searrow \lambda^*$ and $(s_n, x_n, y_n) \in \overline{\Sigma_a^{\lambda_n}}$ such that $\varphi(s_n, x_n, y_n) \geq \varphi^{\lambda_n}(s_n, x_n, y_n)$. However, φ is $L-$periodic in x. Hence, we can assume that $(x_n, y_n) \in \overline{C}$, and consequently, $(s_n, x_n, y_n) \to (\bar{s}, \bar{x}, \bar{y}) \in \overline{\Sigma_a^{\lambda^*}}$. Passing to the limit as $n \to +\infty$, one gets $\varphi(\bar{s}, \bar{x}, \bar{y}) = \varphi^{\lambda^*}(\bar{s}, \bar{x}, \bar{y})$.

Let $z(s,x,y) = \varphi(s,x,y) - \varphi^{\lambda^*}(s,x,y)$ for all $\overline{\Sigma_a^{\lambda^*}}$. The function z is nonpositive and vanishes at the point $(\bar{s},\bar{x},\bar{y})$. Since the equation (3.5.16) is invariant under translation with respect to s and since the function f is Lipschitz-continuous, the nonpositive function z satisfies

$$\begin{cases} \qquad (\tilde{e}A\tilde{e}+\varepsilon)z_{ss} + \nabla_{x,y}\cdot(A\nabla_{x,y}z) + \nabla_{x,y}(A\tilde{e}z_s) \\ +\partial_s(\tilde{e}A\nabla_{x,y}z) + q\cdot\nabla_{x,y}z + (q\cdot\tilde{e}-c)z_s + bz \;=\; 0 \quad \text{in } \Sigma_a^{\lambda^*}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \nu\cdot A\nabla z = 0 \quad \text{on } (-a,-a+\lambda^*)\times\partial\Omega, \end{cases} \tag{3.5.19}$$

for some bounded function $b = b(s,x,y)$. Furthermore,

$$z(-a,x,y) = -\varphi(a-\lambda^*\, x,y) < 0 \;\text{ for all }\; (x,y)\in\overline{\Omega},$$

because φ is continuous and positive on $(-a,a)\times\overline{\Omega}$ and because $\lambda^* < 2a$. Moreover, for all $(x,y)\in\overline{\Omega}$,

$$\begin{aligned} z(-a+\lambda^*,x,y) &= \varphi(-a+\lambda^*,x,y) - \tilde{\varphi}(a,x,y) \\ &\le \varphi(-a+\lambda^*,x,y) - \tilde{\varphi}(-a+\lambda^*,x,y) \;(\tilde{\varphi} \text{ is nondecreasing in } s) \\ &< 0 \end{aligned}$$

after the note mentioned in the beginning of this proof. As a consequence, the point $(\bar{s},\bar{x},\bar{y})$ where z vanishes lies in $(-a,-a+\lambda^*)\times\overline{\Omega}$. But this is ruled out by (3.5.19) and by referring to the strong maximum principle together with Hopf lemma.

Therefore, $\lambda^* = 2a$, and φ is increasing in the variable s in $\overline{\Sigma_a^{\lambda^*}}$.

Let us now turn to the proof of the uniqueness of the solution $\varphi \in C(\overline{\Sigma_a})\cap C^2(\widetilde{\Sigma_a})$ of (3.5.16). Consider two solutions φ and φ'. By arguing as above and sliding φ' with respect to φ, it is found that $\varphi(s,x,y)\le\varphi'(s+2a-\lambda,x,y)$ for all $\lambda\in(0,2a)$ and for all $(s,x,y)\in\overline{\Sigma_a^{\lambda}}$. Passing to the limit $\lambda\to 2a$, one gets $\varphi\le\varphi'$ in $\overline{\Sigma_a}$. On the other hand, sliding φ with respect to φ', it also follows that $\varphi'\le\varphi$ in $\overline{\Sigma_a}$. Eventually, $\varphi\equiv\varphi'$ and the proof of Lemma 3.5.3 is complete. □

Proof of Lemma 3.5.1. The proof of the estimate (3.5.8) was given in details in [1] (Lemma 5.11). In fact, this estimate follows after multiplying (3.5.5) by 1, $\partial_s w^\varepsilon$, and $\partial_{ss}w^\varepsilon$ and then integrating by parts over $[-B,B]\times C$ for any $B>0$. By passing to the limit $B\to+\infty$ in the 3 obtained inequalities, and using the facts $\partial_s w^\varepsilon\to 0$ and $\partial_{ss}w^\varepsilon\to 0$ uniformly in $(x,y)\in\overline{\Omega}$ as $s\to\pm\infty$ together with the $L-$periodicity of the functions w^ε with respect to x, one finally obtains the estimate (3.5.8).

In what follows, we are going to prove the existence of a solution w^ε of (3.5.5) for each $0 < \varepsilon \le \varepsilon_0$ whenever f is of the "ZFK" type. We mention that there is only some little differences in proving the existence of the solution w^ε whenever f is a "combustion" nonlinearity. We will comment, during this present proof, about these differences when they exist.

We consider the regularized problem

$$\varepsilon w^\varepsilon_{ss}(s,x,y) + L_d\, w^\varepsilon(s,x,y) + f(x,y,w^\varepsilon) = 0$$

in cylinders of the type

$$\Sigma_a \;= \{\, (s,x,y),\; -a < s < a,\; (x,y) \in \Omega\},$$

which are bounded in the variable s. One shall then pass to the limit $a \to +\infty$.

Let us fix $a > 0$. Going back to (3.5.4), we call for each $r \in \mathbb{R}$,

$$\psi_r(s,x,y) = \psi(s-r,x,y),$$

for all $(s,x,y) \in \mathbb{R} \times \overline{\Omega}$. It follows from (3.5.4), whose coefficients are independent of s, that ψ_r is a super solution for all $r \in \mathbb{R}$, in the sense that,

$$\varepsilon \partial_{ss}\psi_r(s,x,y) + L_d\, \psi_r(s,x,y) + f(x,y,\psi_r) \le 0.$$

Furthermore, for each $r \in \mathbb{R}$, the function $\psi_r \in E'$ since $\psi \in E'$. In particular, ψ_r is increasing in s and it satisfies $\nu \cdot A(\nabla_{x,y}\psi_r + \tilde{e}\partial_s\psi_r) = 0$ on $\mathbb{R} \times \partial\Omega$.

Let, for all $r \in \mathbb{R}$, h_r be the positive constant defined by

$$0 < h_r := \min_{(x,y)\,\in\,\overline{\Omega}} \psi_r(-a,x,y) \le 1.$$

The constant function h_r clearly satisfies

$$\varepsilon \partial_{ss}h_r(s,x,y) + L_d\, h_r(s,x,y) + f(x,y,h_r) = f(x,y,h_r) \ge 0$$

in $\mathbb{R}\times\overline{\Omega}$, together with $\nu\cdot A(\nabla_{x,y}h_r + \tilde{e}\partial_s h_r) = 0$ on $\mathbb{R}\times\partial\Omega$. Furthermore, $h_r \le \psi_r(s,x,y)$ for all $(s,x,y) \in \overline{\Sigma_a}$ since the function ψ_r is increasing in $s \in \mathbb{R}$.

From the general results of Berestycki and Nirenberg [4] (see also Lemma 5.1 in

[1]), there exists a solution $w_r \in C(\overline{\Sigma_a}) \cap C^2(\overline{\Sigma_a} \setminus \{\pm a\} \times \partial\Omega)$ of

$$\begin{cases} \varepsilon\partial_{ss}w_r + L_d\, w_r + f(x,y,w_r) \;=\; 0 \;\text{ in }\; \Sigma_a \\ \nu \cdot A(\nabla_{x,y}w_r + \tilde{e}\partial_s w_r) = 0 \;\text{ on }\; (-a,a)\times\partial\Omega, \\ w_r \text{ is } L\text{-periodic with respect to } x, \\ w_r(-a,x,y) \;=\; h_r \text{ and } w_r(a,x,y) = \psi_r(a,x,y) \text{ for all } (x,y)\in\overline{\Omega}, \\ 0 < h_r \le w_r(s,x,y) \le \psi_r(s,x,y) \le \psi(s,x,y) \le 1 \text{ for all } (s,x,y)\in\overline{\Sigma_a}. \end{cases} \tag{3.5.20}$$

Moreover, Lemma 3.5.3 yields that w_r is unique and increasing with respect to s. Lastly, the same device as in Lemma 5.3 in [1] yields that w_r is nonincreasing with respect to r, and $r \mapsto w_r$ is continuous with respect to r in $C^{2,\alpha}_{loc}(\overline{\Sigma_a}\setminus\{\pm a\}\times\partial\Omega)$ (for all $0<\alpha<1$) and in $C(\overline{\Sigma_a})$.

Since $0 < h_r \le w_r \le \psi_r \le \psi \le 1$ in $\overline{\Sigma_a}$ and $h_r \to 1$ (resp. $\psi_r \to 0$) as $r \to -\infty$ (resp. $r \to +\infty$) there exists a unique $r_{\varepsilon,a} \in \mathbb{R}$ such that the function $w^{\varepsilon,a} := w_{r_{\varepsilon,a}}$ satisfies (3.5.20) and

$$\max_{(x,y)\in\overline{\Omega}} w^{\varepsilon,a}(0,x,y) = \frac{1}{2}.$$

(when f is a "combustion" nonlinearity, we put the condition $\max_{(x,y)\in\overline{\Omega}} w^{\varepsilon,a}(0,x,y) = \theta$ where θ is the ignition temperature of f, see(3.1.7)).

Fix ε in $(0,\varepsilon_0]$ and consider a sequence $a_n \to +\infty$. From standard elliptic estimates up to the boundary, it follows that the sequence of functions w^{ε,a_n} converge, up to extraction of some subsequence, in $C^{2,\alpha}_{loc}(\mathbb{R}\times\overline{\Omega})$ (for all $0<\alpha<1$) to a function w^ε solving

$$\begin{cases} \varepsilon\partial_{ss}w^\varepsilon + L_d\, w^\varepsilon + f(x,y,w^\varepsilon) \;=\; 0 \;\text{ in }\; \mathbb{R}\times\Omega \\ \nu \cdot A(\nabla_{x,y}w^\varepsilon + \tilde{e}\partial_s w^\varepsilon) = 0 \;\text{ on }\; \mathbb{R}\times\partial\Omega, \\ w^\varepsilon \text{ is } L\text{-periodic with respect to } x, \\ 0 \le w^\varepsilon(s,x,y) \le \psi(s,x,y) \le 1 \text{ for all } (s,x,y)\in\mathbb{R}\times\overline{\Omega}, \end{cases} \tag{3.5.21}$$

together with the condition

$$\max_{(x,y)\in\overline{\Omega}} w^\varepsilon(0,x,y) = \frac{1}{2}. \tag{3.5.22}$$

Furthermore, w^ε is nondecreasing with respect to s.
(In the "combustion" case, (3.5.22) is replaced by $\max_{(x,y)\in\overline{\Omega}} w^\varepsilon(0,x,y) = \theta$).

From the standard elliptic estimates, and from the monotonicity of w^ε with respect to s, it follows that $w^\varepsilon \to \phi_\pm$ in $C^{2,\alpha}(\overline{\Omega})$ as $s \to \pm\infty$ where the functions $\phi_\pm$ satisfy

$$\begin{cases} \nabla\cdot(A\nabla\phi_\pm) + q\cdot\nabla\phi_\pm + f(x,y,\phi_\pm) = 0 \ \text{ in } \ \overline{\Omega}, \\ \nu\cdot A\nabla\phi_\pm = 0 \ \text{ on } \ \partial\Omega, \\ \phi_\pm \ \text{ is } L\text{-periodic with respect to } x, \\ 0 \le \phi_- \ \le \ \phi_+ \le 1. \end{cases} \tag{3.5.23}$$

Integrating by parts over the periodicity cell C leads to (due to (3.1.4))

$$\int_C f(x,y,\phi_\pm)\,dxdy = 0,$$

whence $f(x,y,\phi_\pm) = 0$ for all $(x,y) \in \overline{\Omega}$ by continuity. Now multiply (3.5.23) by $\phi_\pm$ and integrate by parts over C. It follows that

$$\int_\Omega \nabla\phi_\pm\cdot A\nabla\phi_\pm = 0.$$

Owing to (3.1.3), one then gets $\phi_\pm$ are constants. The last inequality of (3.5.21) together with the fact that $\psi(-\infty,x,y) = 0$ uniformly in $(x,y)\in\overline{\Omega}$ ($\psi \in E'$) imply that $\phi_- = 0$ whether in the "ZFK" or in the "combustion" case. Now, if f is a "ZFK" nonlinearity, then $f(x,y,\phi_+) = 0$ yields that $\phi_+ = 0$ or $\phi_+ = 1$. The normalization condition (3.5.22) implies that $\phi_+ = 0$ is impossible, and hence, $\phi_+ = 1$.
(In the "combustion" case, we note that the normalization condition $\max_{(x,y)\in\overline{\Omega}} w^\varepsilon(0,x,y) = \theta$ and the monotonicity of w^ε in s yield that $\phi_+ \ge \theta$. Since $f(x,y,\phi_+) = 0$ in $\overline{\Omega}$, then $\phi_+ = \theta$ or $\phi_+ = 1$. The first case would imply, thanks to the maximum principle, that $w^\varepsilon \equiv \theta$. That is impossible because $w^\varepsilon(-\infty,x,y) = 0$ uniformly in $(x,y)\in\overline{\Omega}$. Eventually, $\phi_+ = 1$ and $w^\varepsilon \to 1$ uniformly in $(x,y)\in\overline{\Omega}$ as $s\to+\infty$).

Let us prove now that, for each $0 < \varepsilon \le \varepsilon_0$, the function w^ε solving (3.5.21) is increasing in $s\in\mathbb{R}$. In fact, it was mentioned above that w^ε is nondecreasing in s. Assume, to the contrary, that there exists $h > 0$ and $(s_0,x_0,y_0)\in\mathbb{R}\times\Omega$ such that $w^\varepsilon(s_0+h,x_0,y_0) = w^\varepsilon(s_0,x_0,y_0)$. Let z be the nonnegative function defined by $z(s,x,y) := w^\varepsilon(s+h,x,y) - w^\varepsilon(s,x,y)$ over $\mathbb{R}\times\overline{\Omega}$. It follows, from (3.5.21), that z satisfies

$$\begin{cases} \varepsilon\partial_{ss}z + L_d\,z + b(s,x,y)z \ = \ 0 \ \text{ in } \ \mathbb{R}\times\Omega, \\ \nu\cdot A(\nabla_{x,y}w^\varepsilon + \tilde{e}\partial_s z) = 0 \ \text{ on } \ \mathbb{R}\times\partial\Omega, \\ z\ge 0 \text{ in } \mathbb{R}\times\overline{\Omega}, \\ z(s_0,x_0,y_0) = 0, \end{cases}$$

for some bounded function $b = b(s, x, y) \in C(\mathbb{R} \times \overline{\Omega})$. The elliptic maximum principle and the Hopf lemma imply that $z \equiv 0$ in $\mathbb{R} \times \overline{\Omega}$. In other words, there exists $h > 0$ such that $w^\varepsilon(s, x, y) = w^\varepsilon(s+h, x, y)$ for all $(s, x, y) \in \mathbb{R}\times\overline{\Omega}$. However, this contradicts with the fact that $w^\varepsilon(-\infty, ., .) = 0$ and $w^\varepsilon(+\infty, ., .) = 1$. Therefore, w^ε is increasing in s and that completes the proof of Lemma 3.5.1. $\square$

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CHAPTER 4

Homogenization and influence of fragmentation in a biological invasion model

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Dedicated to Professor Masayasu Mimura for his 65th birthday

Abstract. In this paper, some properties of the minimal speeds of pulsating Fisher-KPP fronts in periodic environments are established. The limit of the speeds at the homogenization limit is proved rigorously. Near this limit, generically, the fronts move faster when the spatial period is enlarged, but the speeds vary only at the second order. The dependence of the speeds on habitat fragmentation is also analyzed in the case of the patch model.

4.1 Introduction and main hypotheses

In homogeneous environments, the probably most used population dynamics reaction-diffusion model is the Fisher-KPP model [13, 23]. In a one-dimensional space, it corresponds to the following equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u\ (\mu - \nu u),\ t > 0,\ x \in \mathbb{R}. \tag{4.1.1}$$

The unknown $u = u(t, x)$ is the population density at time t and position x, and the positive constant coefficients D, μ and ν respectively correspond to the diffusivity (mobility of the individuals), the intrinsic growth rate and the susceptibility to crowding effects.

A natural extension of this model to heterogeneous environments is the Shigesada-Kawasaki-Teramoto model [32],

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a_L(x)\frac{\partial u}{\partial x}\right) + u\ (\mu_L(x) - \nu_L(x)u),\ t > 0,\ x \in \mathbb{R}, \tag{4.1.2}$$

where the coefficients depend on the space variable x in a $L-$periodic fashion:

Definition 4.1.1 (L-periodicity) *Let L be a positive real number. We say that a function $h : \mathbb{R} \to \mathbb{R}$ is L-periodic if*

$$\forall\ x \in \mathbb{R},\ h(x + L) = h(x).$$

In this paper, we are concerned with the general equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a_L(x)\frac{\partial u}{\partial x}\right) + f_L(x, u), \quad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{4.1.3}$$

The diffusion term a_L satisfies

$$a_L(x) = a(x/L),$$

where a is a $C^{2,\delta}(\mathbb{R})$ (with $\delta > 0$) 1-periodic function that satisfies

$$\exists\ 0 < \alpha_1 < \alpha_2,\ \forall\ x \in \mathbb{R},\ \alpha_1 \le a(x) \le \alpha_2. \tag{4.1.4}$$

On other hand, the reaction term satisfies $f_L(x, \cdot) = f(x/L, \cdot)$, where $f := f(x, s)$: $\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is 1-periodic in x, of class $C^{1,\delta}$ in (x, s) and C^2 in s. In this setting, both a_L and f_L are L-periodic in the variable x. Furthermore, we assume that:

$$\begin{cases} \forall\ x \in \mathbb{R}, \quad f(x, 0) = 0, \\ \exists\ M \ge 0,\ \forall\ s \ge M,\ \forall\ x \in \mathbb{R}, \quad f(x, s) \le 0, \\ \forall\ x \in \mathbb{R}, \quad s \mapsto f(x, s)/s \text{ is decreasing in } s > 0. \end{cases} \tag{4.1.5}$$

Moreover, we set

$$\mu(x) := \lim_{s\to 0^+} f(x,s)/s,$$

and

$$\mu_L(x) := \lim_{s\to 0^+} f_L(x,s)/s = \mu\left(\frac{x}{L}\right).$$

The growth rate μ may be positive in some regions (favorable regions) or negative in others (unfavorable regions).

The stationary states $p(x)$ of (4.1.3) satisfy the equation

$$\frac{\partial}{\partial x}\left(a_L(x)\frac{\partial p}{\partial x}\right) + f_L(x,p) = 0,\ x \in \mathbb{R}. \tag{4.1.6}$$

Under general hypotheses including those of this paper, and in any space dimension, it was proved in [4] that a necessary and sufficient condition for the existence of a positive and bounded solution p of (4.1.6) was the negativity of the principal eigenvalue $\rho_{1,L}$ of the linear operator

$$\mathcal{L}_0 :\ \Phi \mapsto -(a_L(x)\Phi')' - \mu_L(x)\Phi,$$

with periodicity conditions. In this case, the solution p was also proved to be unique, and therefore L-periodic. Actually, it is easy to see that the map $L \mapsto \rho_{1,L}$ is nonincreasing in $L > 0$, and even decreasing as soon as a is not constant (see the proof of Lemma 4.3.1). Furthermore, $\rho_{1,L} \to -\int_0^1 \mu(x)dx$ as $L \to 0^+$. In this paper, in addition to the above-mentioned hypotheses, we make the assumption that

$$\int_0^1 \mu(x)dx > 0. \tag{4.1.7}$$

This assumption then guarantees that

$$\forall\ L > 0, \quad \rho_{1,L} < 0,$$

whence, for all $L > 0$, there exists a unique positive periodic and bounded solution p_L of (4.1.6). Notice that assumption (4.1.7) is immediately fulfilled if $\mu(x)$ is positive everywhere.

In this work, we are concerned with the propagation of pulsating traveling fronts which are particular solutions of the reaction-diffusion equation (4.1.3). Before going further on, we recall the definition of such solutions:

Definition 4.1.2 (Pulsating traveling fronts) *A function $u = u(t,x)$ is called a pulsating traveling front propagating from right to left with an effective speed $c \neq 0$, if* *u is a classical solution of:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(a_L(x)\dfrac{\partial u}{\partial x}\right) + f_L(x,u), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \\ \forall\, k \in \mathbb{Z},\ \forall\,(t,x) \in \mathbb{R} \times \mathbb{R}, \quad u(t + \dfrac{kL}{c}, x) = u(t, x + kL), \\ 0 \le u(t,x) \le p_L(x), \\ \lim\limits_{x \to -\infty} u(t,x) = 0 \text{ and } \lim\limits_{x \to +\infty} u(t,x) - p_L(x) = 0, \end{cases} \tag{4.1.8}$$

where the above limits hold locally in t.

This definition has been introduced in [31, 32]. It has also been extended in higher dimensions with $p_L \equiv 1$ in [1] and [35], and with $p_L \not\equiv 1$ in [5].

Under the above assumptions, it follows from [5] that there exists $c_L^* > 0$ such that pulsating traveling fronts satisfying (4.1.8) with a speed of propagation c exist if and only if $c \ge c_L^*$. Moreover, the pulsating fronts (with speeds $c \ge c_L^*$) are increasing in time t. Further uniqueness and qualitative properties are proved in [14, 15]. The value c_L^* is called the *minimal speed of propagation*. We refer to [2, 3, 11, 18, 25, 27, 28, 34] for further existence results and properties of the minimal speeds of KPP pulsating fronts. For existence, uniqueness, stability and further qualitative results for combustion or bistable nonlinearities in the periodic framework, we refer to [6, 7, 12, 16, 17, 19, 24, 26, 35, 36, 37, 38].

In the particular case of the Shigesada *et al* model (4.1.2), when $a(x) \equiv 1$, the effects of the spatial distribution of the function μ_L on the existence and global stability of a positive stationary state p_L of equation (4.1.2) have been investigated both numerically [30, 31] and theoretically [4, 8, 29]. In particular, as already noticed, enlarging the scale of fragmentation, i.e. increasing L, was proved to decrease the value of $\rho_{1,L}$. Biologically, this result means that larger scales have a positive effect on species persistence, for species whose dynamics is modelled by the Shigesada *et al* model.

The effects of the spatial distribution of the functions a_L and μ_L on the minimal speed of propagation c_L^* have not yet been investigated rigorously. This is a difficult problem, since the known variational formula for c_L^* bears on non-self-adjoint operators, and therefore, the methods used to analyze the dependence of $\rho_{1,L}$ on fragmentation cannot be used in this situation. However, in the case of model (4.1.2), when $a_L \equiv 1$, $\nu_L \equiv 1$ and $\mu_L(x) = \mu(x/L)$, for a 1-periodic function μ taking only two values, Kinezaki *et al* [22] numerically observed that c_L^* was an increasing function of the parameter L. For sinusoidally varying coefficients, the relationships between c_L^* and L have also been investigated formally by Kinezaki, Kawasaki, Shigesada [21]. The case of a rapidly oscillating coefficient $a_L(x)$, corresponding to small L values, and the homogenization limit $L \to 0$, have been discussed in [19] and [38] for combustion and bistable nonlinearities $f(u)$.

The first aim of our work is to analyze rigorously the dependence of the speed of propagation c_L^* with respect to L, under the general setting of equation (4.1.3), for small L values. We determine the limit of the minimal speeds c_L^* as $L \to 0^+$ (the homogenization limit), and we also prove that near the homogenization limit, the species tends to propagate faster when the spatial period of the environment is enlarged. Next, in the case of an environment composed of patches of “habitat" and “non-habitat", we consider the dependence of the minimal speed with respect to habitat fragmentation. We prove that fragmentation decreases the minimal speed.

4.2 Main results

In this section, we describe the main results of this paper. Unless otherwise mentioned, we make the assumptions of Section 4.1. The first theorem gives the limit of c_L^* as L goes to 0.

Theorem 4.2.1 *Let c_L^* be the minimal speed of propagation of pulsating traveling fronts solving* (4.1.8). *Then,*

$$\lim_{L \to 0^+} c_L^* = 2\sqrt{<a>_H <\mu>_A}, \tag{4.2.1}$$

where

$$<\mu>_A = \int_0^1 \mu(x)dx \quad \text{and} \quad <a>_H = \left(\int_0^1 (a(x))^{-1} dx\right)^{-1} = <a^{-1}>_A^{-1}$$

denote the arithmetic mean of μ and the harmonic mean of a over the interval $[0, 1]$.

Formula (4.2.1) was derived formally in [33] for sinusoidally varying coefficients. Theorem 4.2.1 then provides a generalization of the formula in [33] and a rigorous analysis of the homogenization limit for general diffusion and growth rate profiles.

Remark 4.2.2 The previous theorem gives the limit of c_L^* as $L \to 0$ when the space dimension is 1. Theorem 3.3 of El Smaily [11] answered this issue in any dimensions N, but under an additional assumption of free divergence of the diffusion field (in the one-dimensional case considered here, this assumption reduces to $da/dx = 0$ in $\mathbb{R}$). Lastly, we refer to [6, 7, 16] for other homogenization limits with combustion-type nonlinearities.

Our second result describes the behavior of the function $L \mapsto c_L^*$, for small L values.

Theorem 4.2.3 *Let c_L^* be the minimal speed of propagation of pulsating traveling fronts solving* (4.1.8)*. Then, the map $L \mapsto c_L^*$ is of class C^∞ in an interval* $(0, L_0)$ *for some $L_0 > 0$. Furthermore,*

$$\lim_{L\to 0^+} \frac{dc_L^*}{dL} = 0 \tag{4.2.2}$$

and

$$\lim_{L\to 0^+} \frac{d^2 c_L^*}{dL^2} = \gamma \geq 0. \tag{4.2.3}$$

Lastly, $\gamma > 0$ if and only if the function

$$\frac{\mu}{<\mu>_A} + \frac{<a>_H}{a}$$

is not identically equal to 2.

Corollary 4.2.4 *Under the notations of Theorem* 4.2.3*, it follows that if a is constant and μ is not constant, or if μ is constant and a is not constant, then $\gamma > 0$ and the speeds c_L^* are increasing with respect to L when L is close to* 0*.*

Remark 4.2.5 The question of the monotonicity of the map $L \mapsto c_L^*$ had also been studied under different assumptions in [11] (see Theorem 5.3). The author answered this question for a reaction-advection-diffusion equation over a periodic domain $\Omega \subseteq \mathbb{R}^N$, under an additional assumption on the diffusion coefficient (like in Remark 4.2.2, this assumption would mean again in our present setting that the diffusion coefficient $a(x)$ is constant over $\mathbb{R}$). Our result gives the behavior of the minimal speeds of propagation near the homogenization limit for general diffusion and growth rate coefficients. The condition $\gamma > 0$ is generically fulfilled, which means that, roughly speaking, the more oscillating the medium is, the slower the species moves. But the speeds vary only at the second order with respect to the period L. Based on numerical observations which have been carried out in [21] for special types of diffusion and growth rate coefficients, we conjecture that the monotonicity of c_L^* holds for all $L > 0$.

Lastly, we give a first theoretical evidence that habitat fragmentation, without changing the scale L, can decrease the minimal speed c^*. We here fix a period $L_0 > 0$.

We assume that $a \equiv 1$, and that $\mu_{L_0} := \mu_z$ takes only the two values 0 and $m > 0$, and depends on a parameter z. More precisely:

$$\begin{cases} \text{There exist } 0 \leq z \text{ and } l \in (0, L_0) \text{ such that } l + z \leq L_0, \\ \mu_z \equiv m \text{ on } [0, l/2) \cup [l/2 + z, l + z), \\ \mu_z \equiv 0 \text{ on } [l/2, l/2 + z) \cup [l + z, L_0). \end{cases} \tag{4.2.4}$$

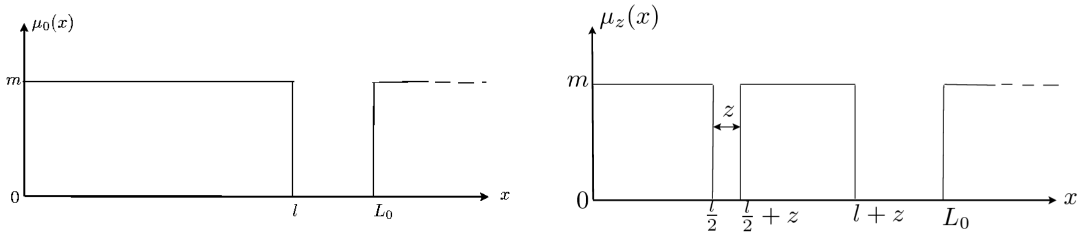

Figure 4.1 – The L_0-periodic function $x \mapsto \mu_z(x)$, (a): with $z = 0$; (b): with $z > 0$.

With this setting, the region where μ_z is positive, which can be interpreted as “habitat" in the Shigesada *et al* model, is of Lebesgue measure l in each period cell $[0, L_0]$. For $z = 0$, this region is simply an interval. However, whenever z is positive, this region is fragmented into two parts of same length $l/2$ (see Figure 4.1). Our next result means that this fragmentation into two parts reduces the speed c^*.

Theorem 4.2.6 *Let c_z^* be the minimal speed of propagation of pulsating traveling fronts solving* (4.1.8)*, with $a_{L_0} \equiv 1$ and $\mu_{L_0} = \mu_z$ defined by* (4.2.4)*. Assume that $l \in (3L_0/4, L_0)$. Then $z \mapsto c_z^*$ is decreasing in $[0, (L_0 - l)/2]$, and increasing in $[(L_0 - l)/2, L_0 - l]$.*

Remark 4.2.7 Note that, whenever $z > (L_0 - l)/2$, the two habitat components in the period cell $[l/2 + z, L_0 + l/2 + z]$ are at a distance smaller than $(L_0 - l)/2$ from each other. In fact, Theorem 4.2.6 proves that, when z varies in $(0, L_0 - l)$, c_z^* is all the larger as the minimal distance separating two habitat components is small, that is as the maximal distance between two consecutive habitat components is large.

Remark 4.2.8 Here, the function μ_z does not satisfy the general regularity assumptions of Section 4.1. However, c_z^* can still be interpreted as the minimal speed of propagation of weak solutions of (4.1.8), whose existence can be obtained by approaching μ_z with regular functions.

The main tool of this paper is a variational formulation for c_L^* involving elliptic eigenvalue problems which depend strongly on the coefficients a and f. Such a formulation was given in any space dimension in [3] in the case where the bounded stationary state p of the equation (4.1.3) is constant, and in [5] in the case of a general nonconstant bounded stationary state $p(x)$.

4.3 The homogenization limit: proof of Theorem 4.2.1

This proof is divided into three main steps.

Step 1: a rough upper bound for c_L^*. For each $L > 0$, the minimal speed c_L^* is positive and, from [5] (see also [3] in the case when $p \equiv 1$), it is given by the variational formula

$$c_L^* = \min_{\lambda>0} \frac{k(\lambda, L)}{\lambda} = \frac{k(\lambda_L^*, L)}{\lambda_L^*}, \tag{4.3.1}$$

where $\lambda_L^* > 0$ and, for each $\lambda \in \mathbb{R}$ and $L > 0$, $k(\lambda, L)$ denotes the principal eigenvalue of the problem

$$\left(a_L \psi'_{\lambda,L}\right)' + 2\lambda a_L \psi'_{\lambda,L} + \lambda a'_L \psi_{\lambda,L} + \lambda^2 a_L \psi_{\lambda,L} + \mu_L \psi_{\lambda,L} = k(\lambda, L)\psi_{\lambda,L} \text{ in } \mathbb{R}, \tag{4.3.2}$$

with L-periodicity conditions. In (4.3.2), $\psi_{\lambda,L}$ denotes a principal eigenfunction, which is of class $C^{2,\delta}(\mathbb{R})$, positive, unique up to multiplication by a positive constant, and L-periodic. Furthermore, it follows from Section 3 of [5] that the map $\lambda \mapsto k(\lambda, L)$ is convex and that $\frac{\partial k}{\partial \lambda}(0, L) = 0$ for each $L > 0$. Therefore, for each $L > 0$, the map $\lambda \mapsto k(\lambda, L)$ is nondecreasing in $\mathbb{R}_+$ and

$$\forall\ \lambda \geq 0,\ \forall\ L > 0, \quad k(\lambda, L) \geq k(0, L) = -\rho_{1,L} > 0 \tag{4.3.3}$$

under the notations of Section 4.1.

Multiplying (4.3.2) by $\psi_{\lambda,L}$ and integrating by parts over $[0, L]$, we get, due to the L-periodicity of a_L and $\psi_{\lambda,L}$:

$$k(\lambda, L) \int_0^L \psi_{\lambda,L}^2 = -\int_0^L a_L \left(\psi'_{\lambda,L}\right)^2 + \lambda^2 \int_0^L a_L \psi_{\lambda,L}^2 + \int_0^L \mu_L \psi_{\lambda,L}^2,$$

for all $\lambda > 0$ and for all $L > 0$. Consequently,

$$\forall\ \lambda > 0,\ \forall\ L > 0, \quad k(\lambda, L) \leq \lambda^2 a_M + \mu_M, \tag{4.3.4}$$

where

$$a_M = \max_{x\in\mathbb{R}} a(x) > 0 \text{ and } \mu_M = \max_{x\in\mathbb{R}} \mu(x) > 0.$$

Using (4.3.1), we get that

$$\forall\ L > 0, \quad 0 < c_L^* \leq 2\sqrt{a_M \mu_M}. \tag{4.3.5}$$

Step 2: the sharp upper bound for c_L^*. For any $\lambda > 0$ and $L > 0$, consider the functions

$$\varphi_{\lambda,L}(x) := e^{\lambda x}\psi_{\lambda,L}(x), \quad x \in \mathbb{R}.$$

Since $\psi_{\lambda,L}$ is unique up to multiplication, we will assume in this step 2 that

$$\int_0^2 \varphi_{\lambda,L}^2(x)dx = 1. \tag{4.3.6}$$

The above choice ensures that

$$\int_0^2 \psi_{\lambda,L}^2(x)dx \leq \int_0^2 e^{2\lambda x}\psi_{\lambda,L}^2(x)dx = \int_0^2 \varphi_{\lambda,L}^2(x)dx = 1. \tag{4.3.7}$$

We are now going to prove that the families $(\psi_{\lambda,L})_{\lambda,L}$ and $(\varphi_{\lambda,L})_{\lambda,L}$ remain bounded in $H^1(0,1)$ for L small enough and as soon as λ stays bounded. For each $L > 0$, we call

$$M_L = [1/L] + 1 \in \mathbb{N},$$

where $[1/L]$ stands for the integer part of $1/L$. Multiplying (4.3.2) by $\psi_{\lambda,L}$ and integrating by parts over $[0, M_L\, L]$, we get that

$$-\int_0^{M_L L} a_L {\psi'_{\lambda,L}}^2 + \int_0^{M_L\, L} \lambda^2 a_L \psi_{\lambda,L}^2 + \int_0^{M_L\, L} \mu_L \psi_{\lambda,L}^2 = k(\lambda, L)\int_0^{M_L L} \psi_{\lambda,L}^2.$$

Using (4.1.4), (4.3.3) and (4.3.4), it follows that

$$0 \leq \int_0^{M_L L} {\psi'_{\lambda,L}}^2 \leq \frac{1}{\alpha_1} \times \left(\lambda^2 a_M + \mu_M\right) \times \int_0^{M_L\, L} \psi_{\lambda,L}^2.$$

Since $1 < M_L L \leq 1 + L$ for all $L > 0$, we have that $1 < M_L L \leq 2$ for all $L \leq 1$. Thus, for all $0 < L \leq 1$,

$$\int_0^1 {\psi'_{\lambda,L}}^2 \leq \int_0^{M_L L} {\psi'_{\lambda,L}}^2 \text{ and } \int_0^{M_L L} \psi_{\lambda,L}^2 \leq \int_0^2 \psi_{\lambda,L}^2 \leq 1$$

from (4.3.7). It follows now that

$$\forall\ \lambda > 0,\ \forall\ 0 < L \leq 1, \quad \int_0^1 {\psi'_{\lambda,L}}^2 \leq \frac{\lambda^2 a_M + \mu_M}{\alpha_1}. \tag{4.3.8}$$

From (4.3.7) and (4.3.8), we conclude that, for any given $\Lambda > 0$, the family $(\psi_{\lambda,L})_{0<\lambda\leq\Lambda,\ 0<L\leq 1}$

is bounded in $H^1(0,1)$. On the other hand,

$$\varphi'_{\lambda,L}(x) = \lambda\varphi_{\lambda,L}(x) + e^{\lambda x}\psi'_{\lambda,L}(x).$$

Owing to (4.3.6) and (4.3.8), we get:

$$\begin{aligned}\forall\ \lambda > 0,\ \forall\ L \leq 1,\ \ ||\varphi'_{\lambda,L}||_{L^2(0,1)} &\leq \lambda \underbrace{||\varphi_{\lambda,L}||_{L^2(0,1)}}_{\leq 1} + e^{\lambda}||\psi'_{\lambda,L}||_{L^2(0,1)} \\ &\leq \lambda + e^{\lambda} \times \sqrt{\frac{\lambda^2 a_M + \mu_M}{\alpha_1}}.\end{aligned} \tag{4.3.9}$$

From (4.3.6) and (4.3.9), we obtain that, for any given $\Lambda > 0$, the family $(\varphi_{\lambda,L})_{0<\lambda\leq\Lambda,\ 0<L\leq 1}$ is bounded in $H^1(0,1)$ and that the family $(a_L\varphi'_{\lambda,L})_{0<\lambda\leq\Lambda,\ 0<L\leq 1}$ is bounded in $L^2(0,1)$ (due to (4.1.4)). Moreover,

$$\left(a_L\varphi'_{\lambda,L}\right)' = \lambda^2 a_L e^{\lambda x}\psi_{\lambda,L} + 2\lambda a_L e^{\lambda x}\psi'_L + \lambda a'_L e^{\lambda x}\psi_{\lambda,L} + e^{\lambda x}a'_L\psi'_{\lambda,L} + e^{\lambda x}a_L\psi''_{\lambda,L}.$$

Multiplying (4.3.2) by $e^{\lambda x}$, we then get

$$\left(a_L\varphi'_{\lambda,L}\right)' + \mu_L\varphi_{\lambda,L} = k(\lambda,L)\varphi_{\lambda,L} \text{ in } \mathbb{R}. \tag{4.3.10}$$

Let

$$v_{\lambda,L}(x) = a_L(x)\varphi'_{\lambda,L}(x)$$

for all $\lambda > 0$, $L > 0$ and $x \in \mathbb{R}$. Pick any $\Lambda > 0$. One already knows that the family $(v_{\lambda,L})_{0<\lambda\leq\Lambda,\ 0<L\leq 1}$ is bounded in $L^2(0,1)$. Furthermore,

$$v'_{\lambda,L} + \mu_L\varphi_{\lambda,L} = k(\lambda,L)\varphi_{\lambda,L} \text{ in } \mathbb{R}. \tag{4.3.11}$$

Notice that the family $(k(\lambda,L))_{0<\lambda\leq\Lambda,\ 0<L\leq 1}$ is bounded from (4.3.3) and (4.3.4). From (4.3.6) and (4.3.11), it follows that the family $\left(v'_{\lambda,L}\right)_{0<\lambda\leq\Lambda,\ 0<L\leq 1}$ is bounded in $L^2(0,1)$. Eventually, $(v_{\lambda,L})_{0<\lambda\leq\Lambda,\ 0<L\leq 1}$ is bounded in $H^1(0,1)$.

Pick now any sequence $(L_n)_{n\in\mathbb{N}}$ such that $0 < L_n \leq 1$ for all $n \in \mathbb{N}$, and $L_n \to 0^+$ as $n \to +\infty$. Choose any $\lambda > 0$ and any sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lambda_n \to \lambda$ as $n \to +\infty$. We claim that

$$k(\lambda_n, L_n) \to \lambda^2 <a>_H + <\mu>_A \ \text{ as } n \to +\infty, \tag{4.3.12}$$

where $<a>_H = \left(\int_0^1 (a(x))^{-1}dx\right)$ and $<\mu>_A = \int_0^1 \mu(x)dx$. To do so, call

$$\psi_n = \psi_{\lambda_n,L_n},\ \varphi_n = \varphi_{\lambda_n,L_n} \text{ and } v_n = v_{\lambda_n,L_n}.$$

It follows from the above computations that the sequences (ψ_n) and (v_n) are bounded in $H^1(0,1)$. Hence, up to extraction of a subsequence,

$$\psi_n \to \overline{\psi} \text{ and } v_n \to w \text{ as } n \to +\infty,$$

strongly in $L^2(0,1)$ and weakly in $H^1(0,1)$. By Sobolev injections, the sequence (ψ_n) is bounded in $C^{0,1/2}([0,1])$. But since each function ψ_n is L_n-periodic (with $L_n \to 0^+$), it follows from Arzela-Ascoli theorem that $\overline{\psi}$ has to be constant over $[0,1]$. Moreover, the boundedness of the sequence $(k(\lambda_n, L_n))_{n\in\mathbb{N}}$ implies that, up to extraction of another subsequence,

$$k(\lambda_n, L_n) \to \overline{k}(\lambda) \in \mathbb{R} \text{ as } n \to +\infty.$$

We denote this limit by $\overline{k}(\lambda)$, we will see later that indeed it depends only on λ. It follows now, from (4.3.11) after replacing (λ, L) by (λ_n, L_n) and passing to the limit as $n \to +\infty$, that

$$w' + <\mu>_A e^{\lambda x}\overline{\psi} = \overline{k}(\lambda)\overline{\psi}e^{\lambda x} \quad \text{a.e. in } (0,1).$$

Notice indeed that $\mu_L \rightharpoonup <\mu>_A$ as $L \to 0^+$ in $L^2(0,1)$ weakly. Meanwhile,

$$\varphi_n' = \lambda_n e^{\lambda_n x}\psi_n + e^{\lambda_n x}\psi_n' = \frac{v_n}{a_{L_n}} \rightharpoonup <a^{-1}>_A\ w \text{ as } n \to +\infty,\ \text{weakly in } L^2(0,1),$$

where $<a^{-1}>_A = \int_0^1 (a(x))^{-1}dx$. Thus, we obtain

$$w = <a^{-1}>_A^{-1}\ \lambda e^{\lambda x}\overline{\psi} = <a>_H\ \lambda e^{\lambda x}\overline{\psi}.$$

Consequently,

$$\lambda^2 <a>_H\ \overline{\psi} + <\mu>_A\ \overline{\psi} = \overline{k}(\lambda)\overline{\psi}.$$

Actually, since the functions ψ_n are L_n-periodic (with $L_n \to 0^+$) and converge to the constant $\overline{\psi}$ strongly in $L^2(0,1)$, they converge to $\overline{\psi}$ in $L^2_{loc}(\mathbb{R})$. But

$$1 = \int_0^2 \varphi_n^2 \le e^{4\lambda_n}\int_0^2 \psi_n^2 \le e^{4M}\int_0^2 \psi_n^2,$$

where $M = \sup_{n\in\mathbb{N}} \lambda_n$. Hence, $\overline{\psi} \neq 0$ and

$$\lambda^2 <a>_H + <\mu>_A = \overline{k}(\lambda). \tag{4.3.13}$$

By uniqueness of the limit, one deduces that the whole sequence $(k(\lambda_n, L_n))_{n\in\mathbb{N}}$ converges to this quantity $\overline{k}(\lambda)$ as $n \to +\infty$, which proves the claim (4.3.12).

Now, take any sequence $L_n \to 0^+$ such that $c^*_{L_n} \to \limsup_{L\to 0^+} c^*_L$ as $n \to +\infty$. For each $\lambda > 0$ and for each $n \in \mathbb{N}$, one has

$$c^*_{L_n} \leq \frac{k(\lambda, L_n)}{\lambda}$$

from (4.3.1), whence

$$\limsup_{L\to 0^+} c^*_L = \lim_{n\to+\infty} c^*_{L_n} \leq \frac{\overline{k}(\lambda)}{\lambda} = \lambda <a>_H + \frac{<\mu>_A}{\lambda}.$$

Since this holds for all $\lambda > 0$, one concludes that

$$\limsup_{L\to 0^+} c^*_L \leq 2\sqrt{<a>_H <\mu>_A}. \tag{4.3.14}$$

Step 3: the sharp lower bound for c^*_L. The aim of this step is to prove that

$$\liminf_{L\to 0^+} c^*_L \geq 2\sqrt{<a>_H <\mu>_A}$$

which would complete the proof of Theorem 4.2.1.

For each $L > 0$, the minimal speed c^*_L is given by (4.3.1) and the map $(0, +\infty) \ni \lambda \mapsto k(\lambda, L)/\lambda$ attains its minimum at $\lambda^*_L > 0$.We will prove that, for L small enough, the family (λ^*_L) is bounded from above and from below by $\overline{\lambda} > 0$ and $\underline{\lambda} > 0$ respectively. Namely, one has

Lemma 4.3.1 *There exist L_0 and $0 < \underline{\lambda} \leq \overline{\lambda} < +\infty$ such that*

$$\underline{\lambda} \leq \lambda^*_L \leq \overline{\lambda} \quad \textit{for all} \quad 0 < L \leq L_0.$$

The proof is postponed at the end of this section. Take now any sequence $(L_n)_n$ such that $0 < L_n \leq L_0$ for all n, and $L_n \to 0^+$ as $n \to +\infty$. From Lemma 4.3.1, there exists $\lambda^* > 0$ such that, up to extraction of a subsequence, $\lambda^*_{L_n} \to \lambda^*$ as $n \to +\infty$. One also has

$$c^*_{L_n} = \frac{k(\lambda^*_{L_n}, L_n)}{\lambda^*_{L_n}} \underset{n\to+\infty}{\to} \frac{\overline{k}(\lambda^*)}{\lambda^*} = \lambda^* <a>_H + \frac{<\mu>_A}{\lambda^*} \geq 2\sqrt{<a>_H <\mu>_A}$$

from (4.3.12) and (4.3.13). Therefore, $\liminf_{L\to 0^+} c^*_L \geq 2\sqrt{<a>_H <\mu>_A}$. Eventually,

$$\lim_{L\to 0^+} c^*_L = 2\sqrt{<a>_H <\mu>_A}$$

and the proof of Theorem 4.2.1 is complete. □

Proof of Lemma 4.3.1. Observe first that, for $\lambda = 0$ and for any $L > 0$, $k(0, L)$ is the principal eigenvalue of the problem

$$(a_L\phi_L')' + \mu_L\phi_L = k(0, L)\phi_L \text{ in } \mathbb{R},$$

and we denote $\phi_L = \psi_{0,L}$ a principal eigenfunction, which is L-periodic, positive and unique up to multiplication. In other words, $k(0, L) = -\rho_{1,L}$ under the notations of Section 4.1. Dividing the above elliptic equation by ϕ_L and integrating by parts over $[0, L]$, one gets

$$k(0, L) = \frac{1}{L}\int_0^L \frac{a_L\, {\phi_L'}^2}{\phi_L^2} + \int_0^1 \mu(x)dx \;\geq <\mu>_A > 0.$$

On the other hand, as already recalled, $\frac{\partial k}{\partial\lambda}(0, L) = 0$ and the map $\lambda \mapsto k(\lambda, L)$ is convex for all $L > 0$. Therefore,

$$\forall\ \lambda > 0,\ \forall\ L > 0,\ k(\lambda, L)\ \geq\ k(0, L)\ \geq <\mu>_A >\ 0.$$

Assume here that there exists a sequence $(L_n)_{n\in\mathbb{N}}$ of positive numbers such that $L_n \to 0^+$ and $\lambda^*_{L_n} \to 0^+$ as $n \to +\infty$. One then gets

$$c^*_{L_n} = \frac{k(\lambda^*_{L_n}, L_n)}{\lambda^*_{L_n}} \geq \frac{<\mu>_A}{\lambda^*_{L_n}} \to +\infty \text{ as } n \to +\infty.$$

This is contradiction with (4.3.14). Thus, for $L > 0$ small enough, the family $(\lambda^*_L)_L$ is bounded from below by a positive constant $\underline{\lambda} > 0$ (actually, these arguments show that the whole family $(\lambda^*_L)_{L>0}$ is bounded from below by a positive constant).

It remains now to prove that $(\lambda^*_L)_L$ is bounded from above when L is small enough. We assume, to the contrary, that there exists a sequence $L_n \to 0^+$ as $n \to +\infty$ such that $\lambda^*_{L_n} \to +\infty$ as $n \to +\infty$. Call

$$k_n = k(\lambda^*_{L_n}, L_n),\ \psi_n(x) = \psi_{\lambda^*_{L_n},L_n}(x) \text{ and } \varphi_n(x) = \varphi_{\lambda^*_{L_n},L_n}(x) = e^{\lambda^*_{L_n}x}\psi_n(x)$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Rewriting (4.3.10) for $\lambda = \lambda^*_{L_n}$ and for $L = L_n$, one consequently gets

$$\forall\, n \in \mathbb{N},\quad (a_{L_n}\varphi_n')' + \mu_{L_n}\varphi_n = k_n\varphi_n \text{ in } \mathbb{R}. \tag{4.3.15}$$

Owing to the positivity and the L_n-periodicity of the $C^2(\mathbb{R})$ eigenfunction ψ_n, it follows that

$$\forall\, n \in \mathbb{N},\quad \exists\, \theta_n \in [0, L_n],\quad \psi_n(\theta_n) = \max_{x\in\mathbb{R}} \psi_n(x) = \max_{x\in[0,L_n]} \psi_n(x),$$

whence

$$\forall n \in \mathbb{N},\quad \psi_n'(\theta_n) = 0.$$

For each $n \in \mathbb{N}$, let $M_{L_n} = [1/L_n] + 1 \in \mathbb{N}$. Thus,

$$\forall\, n \in \mathbb{N},\ \varphi_n'(\theta_n + M_{L_n}L_n) = \lambda^*_{L_n} e^{\lambda^*_{L_n}(\theta_n+M_{L_n}L_n)}\psi_n(\theta_n).$$

Multiplying (4.3.15) by φ_n and integrating by parts over the interval $[\theta_n, \theta_n + M_{L_n}\,L_n]$, one then obtains

$$\begin{array}{c}\underbrace{a_{L_n}(\theta_n + M_{L_n}\,L_n)\varphi_n'(\theta_n + M_{L_n}\,L_n)\varphi_n(\theta_n + M_{L_n}\,L_n) - a_{L_n}(\theta_n)\varphi_n'(\theta_n)\varphi_n(\theta_n)}_{A(n)} \\ - \underbrace{\int_{\theta_n}^{\theta_n+M_{L_n}\,L_n} a_{L_n}{\varphi_n'}^2}_{B(n)} + \underbrace{\int_{\theta_n}^{\theta_n+M_{L_n}\,L_n} \mu_{L_n}\varphi_n^2}_{C(n)} = k_n \int_{\theta_n}^{\theta_n+M_{L_n}\,L_n} \varphi_n^2.\end{array} \tag{4.3.16}$$

But, for each $n \in \mathbb{N}$, $M_{L_n} \in \mathbb{N}$ while a_{L_n} and ψ_n are L_n-periodic. Hence, $a_{L_n}(\theta_n + M_{L_n}\,L_n) = a_{L_n}(\theta_n)$, $\psi_n(\theta_n + M_{L_n}\,L_n) = \psi_n(\theta_n)$, and $\psi_n'(\theta_n + M_{L_n}\,L_n) = \psi_n'(\theta_n) = 0$. Then,

$$\begin{array}{rcl} A(n) & = & a_{L_n}(\theta_n)\lambda^*_{L_n}\psi_n^2(\theta_n)\left(e^{2\lambda^*_{L_n}(\theta_n+M_{L_n}\,L_n)} - e^{2\lambda^*_{L_n}\theta_n}\right) \\ & \geq & \dfrac{\alpha_1}{2} \times \lambda^*_{L_n}\psi_n^2(\theta_n)e^{2\lambda^*_{L_n}(\theta_n+M_{L_n}\,L_n)} \quad (\alpha_1 > 0 \text{ is given by (4.1.4)}), \end{array} \tag{4.3.17}$$

whenever n is large enough so that $2 \leq e^{2\lambda^*_{L_n}M_{L_n}L_n}$ (remember that $\lambda^*_{L_n} \to +\infty$ as

$n \to +\infty$, by assumption). Meanwhile, for all $n \in \mathbb{N}$,

$$|C(n)| \leq \int_{\theta_n}^{\theta_n+M_{L_n} L_n} \left|\mu(\frac{x}{L_n})\right| e^{2\lambda^*_{L_n} x} \psi_n^2(x)dx \leq \mu_\infty \times \frac{\psi_n^2(\theta_n)}{2\lambda^*_{L_n}} \times e^{2\lambda^*_{L_n}(\theta_n+M_{L_n}L_n)}, \tag{4.3.18}$$

where $\mu_\infty = \max_{x\in\mathbb{R}} |\mu(x)|$. On the other hand, (4.3.1) and (4.3.5) yield

$$k_n \leq 2\sqrt{a_M \mu_M} \times \lambda^*_{L_n}$$

for all $n \in \mathbb{N}$, whence

$$\begin{aligned} k_n \int_{\theta_n}^{\theta_n+M_{L_n} L_n} \varphi_n^2 &= k_n \int_{\theta_n}^{\theta_n+M_{L_n} L_n} e^{2\lambda^*_{L_n} x} \psi_n^2(x)dx \\ &\leq \sqrt{a_M \mu_M} \times \psi_n^2(\theta_n) \times e^{2\lambda^*_{L_n}(\theta_n+M_{L_n}L_n)}. \end{aligned} \tag{4.3.19}$$

Now, the term $B(n)$ can be estimated as follows

$$\begin{aligned} B(n) &= \sum_{j=0}^{M_{L_n}-1} \int_{\theta_n+jL_n}^{\theta_n+(j+1)L_n} a_{L_n} e^{2\lambda^*_{L_n} x} \left(\psi_n'(x) + \lambda^*_{L_n}\psi_n(x)\right)^2 dx \\ &\leq \sum_{j=0}^{M_{L_n}-1} \alpha_2\ e^{2\lambda^*_{L_n}(\theta_n+(j+1)L_n)} \int_{\theta_n+jL_n}^{\theta_n+(j+1)L_n} \left(\psi_n'(x) + \lambda^*_{L_n}\psi_n(x)\right)^2 dx \text{ [from (4.1.4)]} \\ &= \sum_{j=0}^{M_{L_n}-1} \alpha_2\ e^{2\lambda^*_{L_n}(\theta_n+(j+1)L_n)} \int_0^{L_n} \left(\psi_n'(x) + \lambda^*_{L_n}\psi_n(x)\right)^2 dx \end{aligned}$$

since ψ_n is L_n-periodic. One has

$$\int_0^{L_n} \left(\psi_n'(x) + \lambda^*_{L_n}\psi_n(x)\right)^2 dx \ \leq\ \psi_n^2(\theta_n) \int_0^{L_n} \left(\frac{\psi_n'(x)}{\psi_n(x)} + \lambda^*_{L_n}\right)^2 dx.$$

We refer now to equation (4.3.2). Taking $\lambda = \lambda^*_{L_n}$, dividing this equation (4.3.2) by the L_n-periodic function ψ_n and then integrating by parts over the interval $[0, L_n]$, we get

$$\int_0^{L_n} a_{L_n} \left(\frac{\psi_n'}{\psi_n}\right)^2 + 2\lambda^*_{L_n} \int_0^{L_n} a_{L_n} \frac{\psi_n'}{\psi_n} + {\lambda^*_{L_n}}^2 \int_0^{L_n} a_{L_n} + \int_0^{L_n} \mu_{L_n} = k_n L_n$$

for all $n \in \mathbb{N}$. Thus,

$$\int_0^{L_n} a_{L_n} \left(\frac{\psi_n'}{\psi_n} + \lambda_{L_n}^* \right)^2 + \underbrace{\int_0^{L_n} \mu_{L_n}}_{>0} = k_n L_n \leq 2\sqrt{a_M \mu_M} \times \lambda_{L_n}^* \, L_n.$$

Owing to (4.1.4), it follows that

$$\forall \; n \in \mathbb{N}, \;\; \int_0^{L_n} \left(\frac{\psi_n'}{\psi_n} + \lambda_{L_n}^* \right)^2 \leq \frac{2\sqrt{a_M \mu_M}}{\alpha_1} \times \lambda_{L_n}^* \, L_n.$$

Putting the above result into $B(n)$, we obtain, for all $n \in \mathbb{N}$,

$$\begin{aligned} B(n) \;\; &\leq \frac{2\alpha_2 \sqrt{a_M \mu_M}}{\alpha_1} \times \lambda_{L_n}^* \, L_n \psi_n^2(\theta_n) \sum_{j=0}^{M_{L_n}-1} e^{2\lambda_{L_n}^*(\theta_n + (j+1)L_n)} \\ &= \frac{2\alpha_2 \sqrt{a_M \mu_M}}{\alpha_1} \times \lambda_{L_n}^* \, L_n \psi_n^2(\theta_n) e^{2\lambda_{L_n}^*(\theta_n + L_n)} \times \frac{e^{2\lambda_{L_n}^* \, L_n M_{L_n}} - 1}{e^{2\lambda_{L_n}^* L_n} - 1} \\ &\leq \frac{2\alpha_2 \sqrt{a_M \mu_M}}{\alpha_1} \times \psi_n^2(\theta_n) \times \frac{\lambda_{L_n}^* \, L_n e^{2\lambda_{L_n}^* L_n}}{e^{2\lambda_{L_n}^* L_n} - 1} \times e^{2\lambda_{L_n}^*(\theta_n + M_{L_n} \, L_n)} \\ &\leq \beta \times \psi_n^2(\theta_n) e^{2\lambda_{L_n}^*(\theta_n + M_{L_n} \, L_n)} \times \left(\lambda_{L_n}^* L_n + 1 \right), \end{aligned} \tag{4.3.20}$$

where $\beta = \left(2\alpha_2 \sqrt{a_M \mu_M} / \alpha_1 \right) \times C$ and C is a positive constant such that

$$\forall x \geq 0, \quad \frac{x e^{2x}}{e^{2x} - 1} \leq C \times (x + 1).$$

Lastly, let us rewrite equation (4.3.16) as

$$\forall \; n \in \mathbb{N}, \;\; A(n) + C(n) - k_n \int_{\theta_n}^{\theta_n + M_{L_n} L_n} \varphi_n^2 = B(n).$$

Together with (4.3.17), (4.3.18), (4.3.19) and (4.3.20), one concludes that there exists

$n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\begin{aligned} &\frac{\alpha_1}{2} \times \lambda^*_{L_n} \psi_n^2(\theta_n) e^{2\lambda^*_{L_n}(\theta_n + M_{L_n} L_n)} \ - \ \mu_\infty \times \frac{\psi_n^2(\theta_n)}{2\lambda^*_{L_n}} \times e^{2\lambda^*_{L_n}(\theta_n + M_{L_n} L_n)} \\ &- \sqrt{a_M \mu_M} \times \psi_n^2(\theta_n) e^{2\lambda^*_{L_n}(\theta_n + M_{L_n} L_n)} \\ &\qquad \leq \ \beta \times \psi_n^2(\theta_n) e^{2\lambda^*_{L_n}(\theta_n + M_{L_n} L_n)} \times \big(\lambda^*_{L_n} L_n + 1\big). \end{aligned} \tag{4.3.21}$$

Divide (4.3.21) by $\lambda^*_{L_n} \psi_n^2(\theta_n) e^{2\lambda^*_{L_n}(\theta_n + M_{L_n} L_n)}$. Then

$$\forall\ n \geq n_0, \quad \frac{\alpha_1}{2} - \frac{\mu_\infty}{2(\lambda^*_{L_n})^2} - \frac{\sqrt{a_M \mu_M}}{\lambda^*_{L_n}} \leq \beta \times \left(L_n + \frac{1}{\lambda^*_{L_n}} \right).$$

Passing to the limit as $n \to +\infty$, one has $L_n \to 0^+$ and $\lambda^*_{L_n} \to +\infty$, whence $\alpha_1 \leq 0$, which is impossible.

Therefore the assumption that $\lambda^*_{L_n} \to +\infty$ as $L_n \to 0^+$ is false and consequently the family $(\lambda^*_L)_L$ is bounded from above by some positive $\overline{\lambda} > 0$ whenever L is small (i.e. $0 < L \leq L_0$). This completes the proof of Lemma 4.3.1. $\square$

Remark 4.3.2 From Theorem 4.2.1, one concludes that the map $(0, +\infty) \ni L \mapsto c^*_L$ can be extended by continuity to the right at $L = 0^+$. Furthermore, for any sequence $(L_n)_n$ of positive numbers such that $L_n \to 0^+$ as $n \to +\infty$, one claims that the positive numbers $\lambda^*_{L_n}$ given in (4.3.1) converge to $\sqrt{<a>_H^{-1} <\mu>_A} = \sqrt{<a^{-1}>_A <\mu>_A}$ as $n \to +\infty$. Indeed

$$\forall\ n \in \mathbb{N}, \quad c^*_{L_n} = \frac{k(\lambda^*_{L_n}, L_n)}{\lambda^*_{L_n}}$$

and Lemma 4.3.1 implies that, up to extraction of a subsequence, $\lambda^*_{L_n} \to \lambda^* > 0$. Passing to the limit as $n \to +\infty$ in the above equation and due (4.3.13) together with Step 2 of the proof of Theorem 4.2.1, one gets

$$2\sqrt{<a>_H <\mu>_A} = \frac{\overline{k}(\lambda^*)}{\lambda^*} = \lambda^* <a>_H + \frac{<\mu>_A}{\lambda^*},$$

whence $\lambda^* = \sqrt{<a>_H^{-1} <\mu>_A}$. Since the limit does not depend on any subsequence, one concludes that the limit of λ^*_L, as $L \to 0^+$, exits and

$$\lim_{L \to 0^+} \lambda^*_L = \sqrt{<a>_H^{-1} <\mu>_A} = \sqrt{<a^{-1}>_A <\mu>_A}.$$

The sharp lower bound of $\liminf_{L\to 0^+} c_L^*$ from the homogenized equation. In the following, we are going to derive the homogenized equation of (4.1.3), which will lead to the sharp lower bound of $\liminf_{L\to 0^+} c_L^*$. However, to furnish this goal we will only consider for the sake of simplicity a particular type of nonlinearities among those satisfying (4.1.5). In fact, the following ideas can be generalized to a wider family of nonlinearities which satisfy (4.1.5), but the proof requires technical extra-arguments which will be the purpose of a forthcoming paper.

For each $L > 0$, let u_L be a pulsating travelling front with minimal speed c_L^* for the reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u_L}{\partial t} = \dfrac{\partial}{\partial x}\left(a_L(x)\dfrac{\partial\, u_L}{\partial x}\right) + \mu(\dfrac{x}{L})g(u_L), \quad t\in\mathbb{R},\ x\in\mathbb{R}, \\ \forall (t,x)\in\mathbb{R}\times\mathbb{R},\ 0 < u_L(t+\dfrac{L}{c_L^*},x) = u_L(t,x+L) < 1, \\ \displaystyle\lim_{x\to-\infty} u_L(t,x) = 0 \text{ and } \lim_{x\to+\infty} u_L(t,x) = 1, \end{cases} \tag{4.3.22}$$

where $a_L(x) = a(x/L)$, a is a $C^{2,\delta}(\mathbb{R})$ 1-periodic function satisfying (4.1.4), μ is a $C^{1,\delta}(\mathbb{R})$ positive 1-periodic function and g is a $C^2(\mathbb{R}_+)$ function such that $g(0) = g(1) = 0$ and $u \mapsto g(u)/u$ is decreasing in $(0,+\infty)$. Up to a shift in time, one can assume that

$$\forall L > 0, \quad \iint_{(0,1)\times(0,1)} u_L(t,x)\,dt\,dx = \frac{1}{2}. \tag{4.3.23}$$

For each $L > 0$, set $f_L(x,u) := f(x/L,u) = \mu(x/L)g(u)$. In this setting, there holds $p_L \equiv 1$. From standard parabolic estimates, each function u_L is (at least) of class $C^2(\mathbb{R}\times\mathbb{R})$. Denote

$$v_L(t,x) = a_L(x)\frac{\partial u_L}{\partial x}(t,x) \text{ and } w_L(t,x) = \frac{\partial u_L}{\partial t}(t,x) \text{ in } \mathbb{R}\times\mathbb{R}.$$

As already underlined, it follows from [1] that $w_L = \frac{\partial u_L}{\partial t} > 0$ in $\mathbb{R}\times\mathbb{R}$ for each $L > 0$. Under the notations of the beginning of this section, it follows from (4.1.4) and (4.3.2) that $k(\lambda, L) \geq \lambda^2\alpha_1 + \mu_m$ for all $L > 0$ and $\lambda\in\mathbb{R}$, where $\mu_m = \min_{\mathbb{R}}\mu > 0$. Hence, $c_L^* \geq 2\sqrt{\alpha_1\mu_m}$ for each $L > 0$ and $\liminf_{L\to 0^+} c_L^* \geq 2\sqrt{\alpha_1\mu_m} > 0$.

We shall now establish some estimates for the functions u_L, v_L and w_L which are independent of L, in order to pass to the limit as $L \to 0^+$. Notice first that standard parabolic estimates and the (t,x)-periodicity satisfied by the functions u_L imply that, for each $L > 0$, $u_L(-\infty,x) = 0$ and $u_L(+\infty,x) = 1$ in $C^2_{loc}(\mathbb{R})$, and $w_L(\pm\infty,x) = 0$ in $C^1_{loc}(\mathbb{R})$.

Let $k \in \mathbb{N}\backslash\{0\}$ be given. Integrating the first equation of (4.3.22) by parts over

$\mathbb{R} \times (-kL, kL)$, one obtains

$$\forall L > 0, \ \iint_{\mathbb{R}\times(-kL,kL)} f(\frac{x}{L}, u_L)\, dt\, dx = 2kL. \tag{4.3.24}$$

Multiplying the first equation of (4.3.22) by u_L and integrating by parts over $\mathbb{R} \times (-kL, kL)$, one then gets

$$\forall L > 0, \ kL = -\iint_{\mathbb{R}\times(-kL,kL)} a_L(x) \left(\frac{\partial u_L}{\partial x}\right)^2 dt\, dx + \iint_{\mathbb{R}\times(-kL,kL)} f(\frac{x}{L}, u_L) u_L\, dt\, dx. \tag{4.3.25}$$

Notice that the last integral in (4.3.25) converges because of (4.3.24) and $0 \le f(x/L, u_L)u_L \le f(x/L, u_L)$. Together with (4.1.4), one concludes that for each $L > 0$, the first integral in (4.3.25) converges and

$$\forall L > 0, \ \iint_{\mathbb{R}\times(-kL,kL)} \left(\frac{\partial u_L}{\partial x}\right)^2 dt\, dx \le \frac{kL}{\alpha_1}.$$

Multiply the first equation of (4.3.22) by $\frac{\partial u_L}{\partial t}$ and integrate by parts over $\mathbb{R}\times(-kL, kL)$. Since

$$\iint_{\mathbb{R}\times(-kL,kL)} \frac{\partial}{\partial x}\left(a_L(x)\frac{\partial u_L}{\partial x}\right)\frac{\partial u_L}{\partial t} = -\frac{1}{2}\int_{\mathbb{R}\times(-kL,kL)} \frac{\partial}{\partial t}\left(a_L(x)\left(\frac{\partial u_L}{\partial x}\right)^2\right) = 0,$$

one obtains that

$$\forall L > 0, \ \iint_{\mathbb{R}\times(-kL,kL)} \left(\frac{\partial u_L}{\partial t}\right)^2 dt\, dx = \int_{(-kL,kL)} F(\frac{x}{L}, 1) dx = 2kL \times \int_0^1 \mu \times \int_0^1 g, \tag{4.3.26}$$

where $F(y, s) = \int_0^s f(y, \tau)d\tau$. It follows from the above estimates that for each compact subset K of $\mathbb{R}$,

$$\forall\, 0 < L < 1, \ \iint_{\mathbb{R}\times K} \left[\left(\frac{\partial u_L}{\partial t}\right)^2 + \left(\frac{\partial u_L}{\partial x}\right)^2\right] dt\, dx \le C(K), \tag{4.3.27}$$

where $C(K)$ is a positive constant depending only on K.

In particular, for each compact K of $\mathbb{R}$ and for each $L > 0$, $||w_L||_{L^2(\mathbb{R}\times K)} \le \sqrt{C(K)}$. Now, differentiate the first equation of (4.3.22) with respect to t (actually, from the regularity of f, the function w_L is of class C^2 with respect to x). There holds

$$\frac{\partial w_L}{\partial t} = \frac{\partial}{\partial x}\left(a_L(x)\frac{\partial\, w_L}{\partial x}\right) + \mu(\frac{x}{L})g'(u_L)w_L \ \text{ in } \mathbb{R}\times\mathbb{R}.$$

Multiply the above equation by w_L and integrate by parts over $\mathbb{R}\times(-kL,kL)$. From (4.1.4) and (4.3.26), it follows that

$$\iint_{\mathbb{R}\times(-kL,kL)} \left(\frac{\partial w_L}{\partial x}\right)^2 dtdx \leq \frac{2kL\eta}{\alpha_1}$$

where η is the positive constant defined by

$$\eta = \max_{x\in\mathbb{R}}\mu(x)\max_{u\in[0,1]}|g'(u)|\max_{x\in\mathbb{R}}|F(x,1)| \geq \frac{1}{2kL}\iint_{\mathbb{R}\times(-kL,kL)}\mu(\frac{x}{L})g'(u_L)w_L^2\,dt\,dx > 0.$$

Then, for each compact $K\subset\mathbb{R}$, there exists a constant $C'(K)>0$ depending only on K such that

$$\forall\, 0<L<1,\ \iint_{\mathbb{R}\times K}\left(\frac{\partial w_L}{\partial x}\right)^2 dt\,dx \leq C'(K). \tag{4.3.28}$$

Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $(0,1)$ such that $L_n\to 0$ and $c^*_{L_n}\to \liminf_{L\to 0^+} c^*_L > 0$ as $n\to+\infty$. It follows from (4.3.27) and the bounds $0<u_{L_n}<1$ that there exists u_0 in $H^1_{loc}(\mathbb{R}\times\mathbb{R})$ such that, up to extraction of a subsequence, $u_{L_n}\to u_0$ strongly in $L^2_{loc}(\mathbb{R}\times\mathbb{R})$ and almost everywhere in $\mathbb{R}\times\mathbb{R}$, and

$$\left(\frac{\partial u_{L_n}}{\partial t},\frac{\partial u_{L_n}}{\partial x}\right) \rightharpoonup \left(\frac{\partial u_0}{\partial t},\frac{\partial u_0}{\partial x}\right) \text{ weakly in } L^2_{loc}(\mathbb{R}\times\mathbb{R}) \text{ as } n\to+\infty.$$

Remember that $v_{L_n} = a_{L_n}\frac{\partial u_{L_n}}{\partial x}$ and $0<\alpha_1\leq a_{L_n}\leq\alpha_2$ for each $n\in\mathbb{N}$. Thus, (4.3.27) yields that for each compact K of $\mathbb{R}$ and for each $n\in\mathbb{N}$, $||v_{L_n}||_{L^2(\mathbb{R}\times K)}\leq\alpha_2 C(K)$. Furthermore, (4.3.22) implies that

$$\forall n\in\mathbb{N},\ \frac{\partial v_{L_n}}{\partial x} = \frac{\partial u_{L_n}}{\partial t} - f(\frac{x}{L_n},u_{L_n}) \text{ in } \mathbb{R}\times\mathbb{R},$$

while $0\leq f(x/L_n,u_{L_n}(t,x))\leq\kappa$ in $\mathbb{R}\times\mathbb{R}$ where $\kappa = \max_{\mathbb{R}}\mu\times\max_{[0,1]}g>0$ is independent of n. Together with (4.3.27), one concludes that the sequence $(\frac{\partial v_{L_n}}{\partial x})_{n\in\mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R}\times\mathbb{R})$. On the other hand, $\frac{\partial v_{L_n}}{\partial t} = a_{L_n}\frac{\partial^2 u_{L_n}}{\partial t\partial x} = a_{L_n}\frac{\partial w_L}{\partial x}$. Owing to (4.1.4) and (4.3.28), the sequence $(\frac{\partial v_{L_n}}{\partial t})_{n\in\mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R}\times\mathbb{R})$. Consequently, up to extraction of another subsequence, there exists $v_0\in H^1_{loc}(\mathbb{R}\times\mathbb{R})$ such that $v_{L_n}\to v_0$

strongly in $L^2_{loc}(\mathbb{R}\times\mathbb{R})$ and

$$\left(\frac{\partial v_{L_n}}{\partial t},\frac{\partial v_{L_n}}{\partial x}\right)\rightharpoonup\left(\frac{\partial v_0}{\partial t},\frac{\partial v_0}{\partial x}\right)\text{ weakly in }L^2_{loc}(\mathbb{R}\times\mathbb{R})\text{ as }n\to+\infty.$$

However, $a^{-1}_{L_n}\rightharpoonup<a^{-1}>_A=<a>^{-1}_H$ in $L^\infty(\mathbb{R})$ weak-$*$ as $n\to+\infty$. Thus,

$$\frac{\partial u_{L_n}}{\partial x}=\frac{v_{L_n}}{a_{L_n}}\rightharpoonup\frac{v_0}{<a>_H}\text{ weakly in }L^2_{loc}(\mathbb{R}\times\mathbb{R})\text{ as }n\to+\infty.$$

By uniqueness of the limit, one gets $v_0=<a>_H\frac{\partial u_0}{\partial x}$. Passing to the limit as $n\to+\infty$ in the first equation of (4.3.22) with $L=L_n$ implies that u_0 is a weak solution of the equation

$$\frac{\partial u_0}{\partial t}=\frac{\partial v_0}{\partial x}+<\mu>_A g(u_0)=<a>_H\frac{\partial^2 u_0}{\partial x^2}+<\mu>_A g(u_0)\text{ in }\mathcal{D}'(\mathbb{R}\times\mathbb{R}).$$

From parabolic regularity, the function u_0 is then a classical solution of the homogenous equation

$$\frac{\partial u_0}{\partial t}=<a>_H\frac{\partial^2 u_0}{\partial x^2}+<\mu>_A\ g(u_0)\text{ in }\mathbb{R}\times\mathbb{R},$$

such that $0\le u_0\le 1$ and $\frac{\partial u_0}{\partial t}\ge 0$ in $\mathbb{R}\times\mathbb{R}$. Lastly, $\iint_{(0,1)^2}u_0(t,x)\,dt\,dx=\frac{1}{2}$ from (4.3.23). On the other hand, it follows from the second equation of (4.3.22) and (4.3.27) that

$$\forall\gamma\in\mathbb{R},\quad u_0(t+\frac{\gamma}{c},x)=u_0(t,x+\gamma)\ \text{ in }\mathbb{R}\times\mathbb{R},$$

where $c=\liminf_{L\to0^+}c^*_L=\lim_{n\to+\infty}c^*_{L_n}>0$. In other words, $u_0(t,x)=U_0(x+ct)$, where U_0 is a classical solution of the equation

$$cU_0'=<a>_H U_0''+<\mu>_A g(U_0),\quad 0\le U_0\le 1\ \text{ in }\mathbb{R}\tag{4.3.29}$$

that satisfies $U_0'\ge 0$ in $\mathbb{R}$ and

$$\int_0^1\left(\int_{cs}^{cs+1}U_0\right)ds=\frac{1}{2}.$$

Standard elliptic estimates imply that U_0 converges as $s\to\pm\infty$ in $C^2_{loc}(\mathbb{R})$ to two constants $U_0^\pm\in[0,1]$ such that $<\mu>_A g(U_0^\pm)=0$, that is $g(U_0^\pm)=0$. The monotonicity of U_0 and the assumption on g imply that $U_0^-=0$ and $U_0^+=1$. In other words, U_0 is a usual travelling front for the homogenized equation (4.3.29) with speed c and limiting conditions 0 and 1 at infinity. Since the minimal speed for this problem is equal to

$2\sqrt{< a >_H < \mu >_A}$, one concludes that

$$\liminf_{L\to 0^+} c_L^* = c \geq 2\sqrt{< a >_H < \mu >_A}.$$

4.4 Monotonicity of the minimal speeds c_L^* near the homogenization limit

This section is devoted to the proof of Theorem 4.2.3. Before going further in the proof, we recall that for each $L > 0$, the minimal speed c_L^* is given by the variational formula

$$c_L^* = \min_{\lambda>0} \frac{k(\lambda, L)}{\lambda} = \frac{k(\lambda_L^*, L)}{\lambda_L^*},$$

where $\lambda_L^* > 0$ and $k(\lambda, L)$ is the principal eigenvalue of the elliptic equation (4.3.2). Notice that $k(\lambda, L)$ can be defined for all $\lambda \in \mathbb{R}$ and $L > 0$.

Step 1: properties of $k(\lambda, L)$ and definition of $\tilde{k}(\lambda, L)$. The principal eigenfunction $\psi_{\lambda,L}$ of (4.3.2) is L-periodic, positive and unique up to multiplication. Denote

$$\phi_{\lambda,L}(x) = \psi_{\lambda,L}(Lx)$$

for all $L > 0$, $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$. Each function $\phi_{\lambda,L}$ is 1-periodic, positive and it is the principal eigenfunction of

$$(a\phi'_{\lambda,L})' + 2L\lambda a\phi'_{\lambda,L} + L\lambda a'\phi_{\lambda,L} + L^2\lambda^2 a\phi_{\lambda,L} + L^2\mu\phi_{\lambda,L} = L^2 k(\lambda, L)\phi_{\lambda,L},$$

associated to the principal eigenvalue $L^2k(\lambda, L)$. But the above problem can be defined for all $\lambda \in \mathbb{R}$ and $L \in \mathbb{R}$. That is, for each $(\lambda, L) \in \mathbb{R}^2$, there exists a unique principal eigenvalue $\tilde{k}(\lambda, L)$ and a unique (up to multiplication) principal eigenfunction $\tilde{\phi}(\lambda, L)$ of

$$(a\tilde{\phi}'_{\lambda,L})' + 2L\lambda a\tilde{\phi}'_{\lambda,L} + L\lambda a'\tilde{\phi}_{\lambda,L} + L^2\lambda^2 a\tilde{\phi}_{\lambda,L} + L^2\mu\tilde{\phi}_{\lambda,L} = \tilde{k}(\lambda, L)\tilde{\phi}_{\lambda,L}. \tag{4.4.1}$$

Furthermore, $\tilde{\phi}_{\lambda,L}$ is 1-periodic, positive and it can be normalized so that

$$\int_0^1 \tilde{\phi}^2_{\lambda,L}(x)dx = 1 \tag{4.4.2}$$

for all $(\lambda, L) \in \mathbb{R}^2$. By uniqueness of the principal eigenelements, it follows that

$$\forall\ L > 0,\ \forall\ \lambda \in \mathbb{R},\quad \tilde{k}(\lambda, L) = L^2 k(\lambda, L)$$

and $\tilde{\phi}_{\lambda,L}$ and $\phi_{\lambda,L}$ are equal up to multiplication by positive constants for each $L > 0$ and $\lambda \in \mathbb{R}$.

Some useful properties of $k(\lambda, L)$ as $L \to 0^+$ shall now be derived from the study the function $\tilde{k}$. Notice first that, since the coefficients of the left-hand side of (4.4.1) are analytic in (λ, L), the function $\tilde{k}$ is analytic, and from the normalization (4.4.2), the functions $\tilde{\phi}_{\lambda,L}$ also depend analytically in $H^2_{loc}(\mathbb{R})$ on the parameters λ and L (see [10, 20]). In particular, the function k is analytic in $\mathbb{R} \times (0, +\infty)$. Observe also that

$$\tilde{k}(\lambda, 0) = 0 \text{ and } \tilde{\phi}_{\lambda,0} = 1 \text{ for all } \lambda \in \mathbb{R}.$$

Lastly, when λ is changed into $-\lambda$ or when L is changed into $-L$, then the operator in (4.4.1) is changed into its adjoint. But since the principal eigenvalues of the operator and its adjoint are identical, it follows that

$$\forall\ (\lambda, L) \in \mathbb{R}^2,\quad \tilde{k}(\lambda, L) = \tilde{k}(\lambda, -L) = \tilde{k}(-\lambda, L).$$

In particular, it follows that

$$\forall\ (i,j) \in \mathbb{N}^2,\quad \frac{\partial^i \tilde{k}}{\partial \lambda^i}(\lambda, 0) = \frac{\partial^i \partial^{2j+1} \tilde{k}}{\partial \lambda^i \partial L^{2j+1}}(\lambda, 0) = 0. \tag{4.4.3}$$

Therefore, for all $\overline{\lambda} \in \mathbb{R}$,

$$k(\lambda, L) = \frac{\tilde{k}(\lambda, L)}{L^2} \to \frac{1}{2} \times \frac{\partial^2 \tilde{k}}{\partial L^2}(\overline{\lambda}, 0) \text{ as } (\lambda, L) \to (\overline{\lambda}, 0^+).$$

But since this limit is equal to $\overline{k}(\overline{\lambda}) = \overline{\lambda}^2 <a>_H + <\mu>_A$ from Step 2 of the proof of Theorem 4.2.1, one then gets that

$$\frac{1}{2} \times \frac{\partial^2 \tilde{k}}{\partial L^2}(\overline{\lambda}, 0) = \overline{\lambda}^2 <a>_H + <\mu>_A \text{ for all } \overline{\lambda} \in \mathbb{R}. \tag{4.4.4}$$

It also follows from (4.4.3) that

$$\frac{\partial^2 k}{\partial \lambda^2}(\lambda, L) = \frac{1}{L^2} \times \frac{\partial^2 \tilde{k}}{\partial \lambda^2}(\lambda, L) \to \frac{1}{2} \times \frac{\partial^4 \tilde{k}}{\partial \lambda^2 \partial L^2}(\overline{\lambda}, 0) \text{ as } (\lambda, L) \to (\overline{\lambda}, 0^+). \tag{4.4.5}$$

From (4.4.4) and (4.4.5), one deduces that

$$\frac{\partial^2 k}{\partial \lambda^2}(\lambda, L) \to 2 <a>_H \ > 0 \text{ as } (\lambda, L) \to (\overline{\lambda}, 0^+). \tag{4.4.6}$$

Similarly, as $(\lambda, L) \to (\overline{\lambda}, 0^+)$,

$$\left\{ \begin{aligned} \frac{\partial k}{\partial L}(\lambda, L) = \frac{\partial}{\partial L}\left(\frac{\tilde{k}(\lambda, L)}{L^2}\right) &\to \frac{1}{6} \times \frac{\partial^3 \tilde{k}}{\partial L^3}(\overline{\lambda}, 0) = 0 \\ \frac{\partial^2 k}{\partial \lambda \partial L}(\lambda, L) = \frac{\partial}{\partial L}\left(\frac{1}{L^2} \times \frac{\partial \tilde{k}}{\partial \lambda}(\lambda, L)\right) &\to \frac{1}{6} \times \frac{\partial^4 \tilde{k}}{\partial \lambda \partial L^3}(\overline{\lambda}, 0) = 0 \\ \frac{\partial^2 k}{\partial L^2}(\lambda, L) = \frac{\partial^2}{\partial L^2}\left(\frac{\tilde{k}(\lambda, L)}{L^2}\right) &\to \frac{1}{12} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\overline{\lambda}, 0) \end{aligned} \right. \tag{4.4.7}$$

Remark 4.4.1 As a byproduct of the fact that $\tilde{k}$ and k are even in λ, it follows that the minimal speed of pulsating fronts propagating from right to left (as in Definition 4.1.2) is the same as that of fronts propagating from left to right.

Step 2: properties of c^*_L and λ^*_L in the neighbourhood of $L = 0^+$. Let us first prove that, for each fixed $L > 0$, the positive real number $\lambda^*_L > 0$ given in (4.3.1) is unique. Indeed, if there are $0 < \lambda_1 < \lambda_2$ such that

$$c^*_L = \frac{k(\lambda_1, L)}{\lambda_1} = \frac{k(\lambda_2, L)}{\lambda_2} = \min_{\lambda > 0} \frac{k(\lambda, L)}{\lambda},$$

then $k(\lambda, L) = c^*_L \lambda$ for all $\lambda \in [\lambda_1, \lambda_2]$ since k is convex with respect to λ. Then $k(\lambda, L) = c^*_L \lambda$ for all $\lambda \in \mathbb{R}$ by analyticity of the map $\mathbb{R} \ni \lambda \mapsto k(\lambda, L)$. But $k(0, L) = -\rho_{1,L} > 0$, which gives a contradiction. Therefore, for each $L > 0$, λ^*_L is the unique minimum of the map $(0, +\infty) \ni \lambda \mapsto k(\lambda, L)/\lambda$.

Furthermore, we claim that $L \mapsto \lambda^*_L$ and $L \mapsto c^*_L$ are of class C^∞ in a right neighbourhood of $L = 0$. Indeed, by definition, λ^*_L satisfies

$$F(\lambda^*_L, L) := \frac{\partial k}{\partial \lambda}(\lambda^*_L, L) \times \lambda^*_L - k(\lambda^*_L, L) = 0. \tag{4.4.8}$$

The function $(\lambda, L) \mapsto F(\lambda, L)$ is of class C^∞ on $\mathbb{R} \times (0, +\infty)$ and $\frac{\partial F}{\partial \lambda}(\lambda, L) = \frac{\partial^2 k}{\partial \lambda^2}(\lambda, L) \times \lambda$. But

$$\lambda^*_L \to \lambda^* = \sqrt{<a>_H^{-1} <\mu>_A} \ > \ 0 \text{ as } L \to 0^+$$

from Remark 4.3.2, and

$$\frac{\partial^2 k}{\partial \lambda^2}(\lambda_L^*, L) \ \to \ 2 \ <a>_H \ > 0 \text{ as } L \to 0^+$$

from (4.4.6). Therefore, from the implicit function theorem, the map $L \mapsto \lambda_L^*$ is of class C^∞ in an interval $(0, L_0)$ for some $L_0 > 0$. As a consequence of formula (4.3.1), the map $L \mapsto c_L^*$ is also of class C^∞ on $(0, L_0)$.

For each $L \in (0, L_0)$, one has

$$\frac{dc_L^*}{dL} = \left(\frac{1}{\lambda_L^*} \times \frac{\partial k}{\partial \lambda}(\lambda_L^*, L) - \frac{k(\lambda_L^*, L)}{(\lambda_L^*)^2}\right) \times \frac{d\lambda_L^*}{dL} + \frac{1}{\lambda_L^*} \times \frac{\partial k}{\partial L}(\lambda_L^*, L) = \frac{1}{\lambda_L^*} \times \frac{\partial k}{\partial L}(\lambda_L^*, L)$$

by definition of λ_L^* and formula (4.3.1). But $\lambda_L^* \to \lambda^* > 0$ and $\frac{\partial k}{\partial L}(\lambda_L^*, L) \to 0$ as $L \to 0^+$ from (4.4.7). Thus,

$$\frac{dc_L^*}{dL} \to 0 \text{ as } L \to 0^+.$$

On the other hand, it follows from (4.4.6), (4.4.7) and (4.4.8) that

$$\frac{d\lambda_L^*}{dL} = \frac{1}{\lambda_L^* \times \frac{\partial^2 k}{\partial \lambda^2}(\lambda_L^*, L)} \times \left(\frac{\partial k}{\partial L}(\lambda_L^*, L) - \lambda_L^* \times \frac{\partial^2 k}{\partial \lambda \partial L}(\lambda_L^*, L)\right) \to 0 \text{ as } L \to 0^+.$$

Therefore,

$$\begin{aligned} \frac{d^2 c_L^*}{dL^2} &= \frac{d\lambda_L^*}{dL} \times \left(-\frac{1}{(\lambda_L^*)^2} \times \frac{\partial k}{\partial L}(\lambda_L^*, L) + \frac{1}{\lambda_L^*} \times \frac{\partial^2 k}{\partial \lambda \partial L}(\lambda_L^*, L)\right) + \frac{1}{\lambda_L^*} \times \frac{\partial^2 k}{\partial L^2}(\lambda_L^*, L) \\ &\to \frac{1}{12\lambda^*} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) \ \text{ as } L \to 0^+, \end{aligned} \tag{4.4.9}$$

from (4.4.7).

<u>Step 3: calculation of $\frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0)$.</u> In this step, we fix $\lambda^* = \sqrt{<a>_H^{-1} <\mu>_A}$. Since the functions $\tilde{\phi}_{\lambda^*, L}$ depend analytically on $L \in \mathbb{R}$ in $H^2_{loc}(\mathbb{R})$, the expansion

$$\tilde{\phi}_{\lambda^*, L} = 1 + L\phi_1 + L^2\phi_2 + L^3\phi_3 + L^4\phi_4 + \ldots$$

is valid in $H^2_{loc}(\mathbb{R})$ in a neighbourhood of $L = 0$, where $1 = \tilde{\phi}_{\lambda^*, 0}$ and

$$\phi_i = \frac{1}{i\,!} \times \left.\frac{\partial^i \tilde{\phi}_{\lambda^*, L}}{\partial L^i}\right|_{L=0}$$

for each $i \geq 1$. We now put this expansion into

$$(a\tilde{\phi}'_{\lambda^*,L})' + 2L\lambda^* a\tilde{\phi}'_{\lambda^*,L} + L\lambda^* a'\tilde{\phi}_{\lambda^*,L} + L^2(\lambda^*)^2 a\tilde{\phi}_{\lambda^*,L} + L^2\mu\tilde{\phi}_{\lambda^*,L} = \tilde{k}(\lambda^*, L)\tilde{\phi}_{\lambda^*,L}$$

and remember that

$$\tilde{k}(\lambda^*, 0) = \frac{\partial \tilde{k}}{\partial L}(\lambda^*, 0) = \frac{\partial^3 \tilde{k}}{\partial L^3}(\lambda^*, 0) = 0$$

and

$$\frac{\partial^2 \tilde{k}}{\partial L^2}(\lambda^*, 0) = 2 \times \left[(\lambda^*)^2 <a>_H + <\mu>_A\right] = 4 <\mu>_A$$

from (4.4.3) and (4.4.4). Since both $\tilde{\phi}_{\lambda^*,L}$ and $\tilde{k}(\lambda^*, L)$ depend analytically on L, it follows in particular that

$$\begin{cases} (a\phi_1')' + \lambda^* a' = 0, \\ (a\phi_2')' + 2\lambda^* a\phi_1' + \lambda^* a'\phi_1 + (\lambda^*)^2 a + \mu = 2 <\mu>_A, \\ (a\phi_3')' + 2\lambda^* a\phi_2' + \lambda^* a'\phi_2 + (\lambda^*)^2 a\phi_1 + \mu\phi_1 = 2 <\mu>_A \phi_1, \\ (a\phi_4')' + 2\lambda^* a\phi_3' + \lambda^* a'\phi_3 + (\lambda^*)^2 a\phi_2 + \mu\phi_2 = 2 <\mu>_A \phi_2 + \dfrac{1}{24} \times \dfrac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) \end{cases} \tag{4.4.10}$$

in $\mathbb{R}$. Furthermore, each function ϕ_i is 1-periodic and, by differentiating the normalization condition $\|\tilde{\phi}_{\lambda^*,L}\|^2_{L^2(0,1)} = 1$ with respect to L at $L = 0$, it follows especially that

$$\int_0^1 \phi_1 = 0 \text{ and } \int_0^1 \phi_2 = -\frac{1}{2}\int_0^1 \phi_1^2.$$

It is then found that, for all $x \in \mathbb{R}$,

$$a\phi_1'(x) = -\lambda^* a(x) + \lambda^* <a>_H, \tag{4.4.11}$$

$$\phi_1(x) = \lambda^* \times \left(-x + <a>_H \int_0^x \frac{1}{a(y)}dy - \frac{1}{2} + <a>_H \int_0^1 \frac{y}{a(y)}dy\right) \tag{4.4.12}$$

and

$$\begin{aligned} \phi_2'(x) \;&=\; <\mu>_A \times \left[\frac{x}{a(x)} - \int_0^1 \frac{y}{a(y)}dy - \int_0^x \frac{1}{a(y)}dy - \frac{<a>_H}{a(x)}\int_0^1 \frac{y}{a(y)}dy\right] \\ &\quad + \frac{1}{a(x)} \times \left[<a>_H \int_0^1 \left(\frac{1}{a(y)}\int_0^y \mu(z)dz\right)dy - \int_0^x \mu(y)dy\right] + (\lambda^*)^2\left(x + \frac{1}{2}\right). \end{aligned}$$

Moreover, it follows from the third equation of (4.4.10) that, for all $x \in \mathbb{R}$,

$$\begin{aligned} a(x)\phi_3'(x) &= -2\lambda^* \int_0^x a(y)\phi_2'(y)dy - \lambda^* \int_0^x a'(y)\phi_2(y)dy - (\lambda^*)^2 \int_0^x a(y)\phi_1(y)dy \\ &\quad - \int_0^x \mu(y)\phi_1(y)dy + 2 <\mu>_A \int_0^x \phi_1(y)dy + <a>_H c, \end{aligned} \tag{4.4.13}$$

where

$$\begin{aligned} c = \int_0^1 \Bigg[\frac{1}{a(y)} \times \Bigg(2\lambda^* \int_0^y a(z)\phi_2'(z)dz + \lambda^* \int_0^y a'(z)\phi_2(z)dz + (\lambda^*)^2 \int_0^y a(z)\phi_1(z)dz \\ + \int_0^y \mu(z)\phi_1(z)dz - 2 <\mu>_A \int_0^y \phi_1(z)dz \Bigg) \Bigg] dy. \end{aligned} \tag{4.4.14}$$

On the other hand, by integrating the fourth equation of (4.4.10) over the interval $[0,1]$, one gets that

$$\frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) = -2 <\mu>_A \int_0^1 \phi_2 + \lambda^* \int_0^1 a\phi_3' + (\lambda^*)^2 \int_0^1 a\phi_2 + \int_0^1 \mu\phi_2. \tag{4.4.15}$$

One multiplies the second equation of (4.4.10) by ϕ_2 and integrates by parts over $[0,1]$ to get

$$2 <\mu>_A \int_0^1 \phi_2 = -\int_0^1 a{\phi_2'}^2 + 2\lambda^* \int_0^1 a\phi_1'\phi_2 + \lambda^* \int_0^1 a'\phi_1\phi_2 + \lambda^{*2} \int_0^1 a\phi_2 + \int_0^1 \mu\phi_2.$$

Now, putting the above equation into (4.4.15) and integrating by parts, it follows that

$$\frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) = \lambda^* \int_0^1 a\phi_3' + \int_0^1 a{\phi_2'}^2 - \lambda^* \int_0^1 a\phi_1'\phi_2 + \lambda^* \int_0^1 a\phi_1\phi_2'. \tag{4.4.16}$$

To develop $\int_0^1 a\phi_3'$, we use the fact that, for any continuous function G,

$$\int_0^1 \int_0^x G(y)dydx = \int_0^1 (1-x)G(x)dx.$$

Hence, it follows from (4.4.13) that

$$\begin{aligned}\int_0^1 a\phi_3' = & -\lambda^* \int_0^1 a\phi_2' + \lambda^* \int_0^1 xa(x)\phi_2'(x)dx + \lambda^* a(0)\phi_2(0) - \lambda^* \int_0^1 a\phi_2 \\ & -\lambda^{*2} \int_0^1 a\phi_1 + \lambda^{*2} \int_0^1 xa\phi_1 - \int_0^1 \mu\phi_1 + \int_0^1 x\mu\phi_1 \\ & -2 <\mu>_A \int_0^1 x\phi_1 + c <a>_H .\end{aligned} \quad (4.4.17)$$

In what follows, we denote

$$B(x) = \int_0^x \frac{1}{a(y)}dy \quad \text{and} \quad D(x) = \int_0^x \mu(y)dy.$$

Since for any continuous function h

$$\int_0^1 \frac{1}{a(y)} \int_0^y h(z)dzdy = \frac{1}{<a>_H} \int_0^1 h(z)dz - \int_0^1 h(z)B(z)dz,$$

we get that

$$\int_0^1 \frac{1}{a(y)} \left(\int_0^y a(z)\phi_2'(z)dz \right) dy = \frac{1}{<a>_H} \int_0^1 a\phi_2' - \int_0^1 aB\phi_2'.$$

One obtains from (4.4.14) that

$$\begin{aligned}\lambda^* c <a>_H = & \lambda^{*2} \int_0^1 a\phi_2' - \lambda^{*2} <a>_H \int_0^1 aB\phi_2' + \lambda^{*2} <a>_H \int_0^1 \phi_2 \\ & -\lambda^{*2} a(0)\phi_2(0) + \lambda^{*3} \int_0^1 a\phi_1 - \lambda^{*3} <a>_H \int_0^1 aB\phi_1 + \lambda^* \int_0^1 \mu\phi_1 \\ & -\lambda^* <a>_H \int_0^1 \mu B\phi_1 + 2\lambda^* < \mu >_A <a>_H \int_0^1 B\phi_1.\end{aligned} \quad (4.4.18)$$

Putting (4.4.17) and (4.4.18) into (4.4.16), it follows that

$$\begin{aligned}\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0) = \ & \int_0^1 a{\phi_2'}^2 - \lambda^*\int_0^1 a\phi_1'\phi_2 + \lambda^*\int_0^1 a\phi_1\phi_2' + \lambda^{*2}\int_0^1 xa\phi_2' \\ & -\lambda^{*2}\int_0^1 a\phi_2 + \lambda^{*3}\int_0^1 xa\phi_1 + \lambda^*\int_0^1 x\mu\phi_1 \\ & -2\lambda^* <\mu>_A\int_0^1 x\phi_1 - \lambda^{*2} <a>_H\int_0^1 Ba\phi_2' \\ & +\lambda^{*2} <a>_H\int_0^1 \phi_2 - \lambda^{*3} <a>_H\int_0^1 Ba\phi_1 \\ & -\lambda^* <a>_H\int_0^1 B\mu\phi_1 + 2\lambda^* <\mu>_A<a>_H\int_0^1 B\phi_1.\end{aligned}$$

Referring to (4.4.11) and (4.4.12) one obtains

$$\begin{aligned}\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0) = \ & \int_0^1 a{\phi_2'}^2 + \lambda^{*3}\int_0^1 xa\phi_1 - \lambda^{*3} <a>_H\int_0^1 Ba\phi_1 - \frac{\lambda^{*2}}{2}\int_0^1 a\phi_2' \\ & +\lambda^{*2} <a>_H<\frac{x}{a}>_A\int_0^1 a\phi_2' + \lambda^*\int_0^1 x\mu\phi_1 \\ & -\lambda^* <a>_H\int_0^1 B\mu\phi_1 + 2\lambda^* <\mu>_A<a>_H\int_0^1 B\phi_1 \\ & -2\lambda^* <\mu>_A\int_0^1 x\phi_1.\end{aligned} \tag{4.4.19}$$

Moreover,

$$\begin{aligned}\int_0^1 a\phi_2' = \ & <\mu>_A - \int_0^1 D(x)dx - \frac{\lambda^{*2}}{2} <a>_H - \lambda^{*2} <a>_H\int_0^1 aB \\ & +\frac{\lambda^{*2}}{2}\int_0^1 a + \lambda^{*2}\int_0^1 xa - \lambda^{*2} <a>_H<\frac{x}{a}>_A\int_0^1 a \\ & + <a>_H\int_0^1\frac{D}{a} - 2 <\mu>_A<a>_H<\frac{x}{a}>_A + \lambda^{*2} <a>_H^2<\frac{x}{a}>_A,\end{aligned}$$

where

$$\int_0^1 D(x)dx = \int_0^1\int_0^x \mu(y)dydx = \ <\mu>_A - \int_0^1 x\mu, \tag{4.4.20}$$

and

$$\begin{aligned}\int_0^1 \frac{D(x)}{a(x)}dx &= \int_0^1 \int_y^1 \frac{1}{a(x)}\mu(y)dxdy \\ &= \int_0^1 \mu(y)\left(\frac{1}{<a>_H} - B(y)\right)dy = \frac{<\mu>_A}{<a>_H} - \int_0^1 \mu B.\end{aligned} \tag{4.4.21}$$

Having the above two equations together with the fact that $\lambda^{*2} =<\mu>_A<a>_H^{-1}$, one then gets

$$\begin{aligned}\int_0^1 a\phi_2' = \; & \frac{<\mu>_A}{2} + \int_0^1 x\mu - <\mu>_A<a>_H<\frac{x}{a}>_A - <a>_H \int_0^1 B\mu \\ & - <\mu>_A \int_0^1 Ba - <\mu>_A<\frac{x}{a}>_A \int_0^1 a + \frac{<\mu>_A}{<a>_H}\int_0^1 xa \\ & + \frac{<\mu>_A}{2<a>_H}\int_0^1 a.\end{aligned} \tag{4.4.22}$$

Putting (4.4.11), (4.4.12) and (4.4.22) into (4.4.19) and using $\lambda^{*2} =<\mu>_A<a>_H^{-1}$, one then gets

$$\begin{aligned}
\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0) = \;& \int_0^1 a{\phi_2'}^2 + 2 < \mu >_A^2 < a >_H \int_0^1 B^2 - < \mu >_A^2 \int_0^1 B^2 a \\
& - < \mu >_A < a >_H \int_0^1 B^2\mu + 2\frac{< \mu >_A^2}{< a >_H}\int_0^1 xBa \\
& -2 < \mu >_A^2 < \frac{x}{a} >_A \int_0^1 Ba \;+\frac{< \mu >_A^2}{< a >_H}\int_0^1 Ba \\
& +2 < \mu >_A \int_0^1 xB\mu + < \mu >_A \int_0^1 B\mu \\
& -2 < \mu >_A < a >_H < \frac{x}{a} >_A \int_0^1 B\mu - 4 < \mu >_A^2 \int_0^1 xB \\
& - < \mu >_A^2 \int_0^1 B + 2 < \mu >_A^2 < a >_H < \frac{x}{a} >_A \int_0^1 B \\
& -\frac{< \mu >_A}{< a >_H}\int_0^1 x^2\mu - \frac{< \mu >_A}{< a >_H}\int_0^1 x\mu \\
& +2 < \mu >_A < \frac{x}{a} >_A \int_0^1 x\mu - \frac{< \mu >_A^2}{< a >_H^2}\int_0^1 x^2 a - \frac{< \mu >_A^2}{< a >_H^2}\int_0^1 xa \\
& +2\frac{< \mu >_A^2}{< a >_H} < \frac{x}{a} >_A \int_0^1 xa \\
& -\frac{< \mu >_A^2}{4 < a >_H}\Big(2 < a >_H < \frac{x}{a} >_A -1\Big)^2 + \frac{2 < \mu >_A^2}{3 < a >_H} + \frac{< \mu >_A^2}{2 < a >_H} \\
& -\frac{< \mu >_A^2}{4 < a >_H}\Big(2 < a >_H < \frac{x}{a} >_A -1\Big)^2 \int_0^1 a - < \mu >_A^2 < \frac{x}{a} >_A .
\end{aligned}$$

Moreover, for each $x\in\mathbb{R}$, one can rewrite $\phi_2'(x)$ as

$$\begin{aligned}
\phi_2'(x) = \;& < \mu >_A \frac{x}{a(x)} - \frac{D(x)}{a(x)} + \frac{< \mu >_A}{< a >_H}x - < \mu >_A B(x) + \frac{< \mu >_A}{2 < a >_H} \\
& - < \mu >_A < \frac{x}{a} >_A + \frac{< \mu >_A}{a} - \frac{< a >_H}{a}\int_0^1 B\mu - \frac{< \mu >_A < a >_H < \frac{x}{a} >_A}{a}.
\end{aligned}$$

Consequently, the expanded form of $a(x)(\phi_2'(x))^2$ consists of 45 terms. One integrates the expanded form of $x \mapsto a(x)\phi_2'^2(x)$ over the interval $[0,1]$ to obtain

$$
\begin{aligned}
\int_0^1 a{\phi_2'}^2 = \; & <\mu>_A^2 \int_0^1 aB^2 + <a>_H \left(\int_0^1 B\mu\right)^2 - 2<\mu>_A^2 \int_0^1 xB \\
& +2<\mu>_A \int_0^1 BD - 2\frac{<\mu>_A^2}{<a>_H}\int_0^1 xaB \\
& -\frac{<\mu>_A^2}{<a>_H}\int_0^1 aB + 2<\mu>_A^2<\frac{x}{a}>_A \int_0^1 aB \\
& -2<\mu>_A^2\int_0^1 B + 2<\mu>_A^2<a>_H\int_0^1 B\int_0^1 B\mu \\
& +2<\mu>_A^2<a>_H<\frac{x}{a}>_A\int_0^1 B - 2<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1 B\mu \\
& +2<a>_H\int_0^1\frac{D}{a}\int_0^1 B\mu - <\mu>_A\int_0^1 B\mu - <\mu>_A\int_0^1 B\mu \\
& +2<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1 B\mu - 2<\mu>_A\int_0^1 B\mu \\
& +2<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1 B\mu + <\mu>_A^2\int_0^1\frac{x^2}{a} - 2<\mu>_A\int_0^1\frac{xD}{a} \\
& +\frac{2<\mu>_A^2}{3<a>_H} + \frac{<\mu>_A^2}{2<a>_H} - <\mu>_A^2<\frac{x}{a}>_A + 2<\mu>_A^2<\frac{x}{a}>_A \\
& -2<\mu>_A^2<a>_H<\frac{x}{a}>_A^2 + \int_0^1\frac{D^2}{a} - 2\frac{<\mu>_A}{<a>_H}\int_0^1 xD - \frac{<\mu>_A}{<a>_H}\int_0^1 D \\
& +2<\mu>_A<\frac{x}{a}>_A\int_0^1 D - 2<\mu>_A\int_0^1\frac{D}{a} + 2<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1\frac{D}{a} \\
& +\frac{<\mu>_A^2}{<a>_H^2}\int_0^1 ax^2 + \frac{<\mu>_A^2}{<a>_H^2}\int_0^1 ax - 2\frac{<\mu>_A^2}{<a>_H}<\frac{x}{a}>_A\int_0^1 ax + \frac{<\mu>_A^2}{<a>_H} \\
& -<\mu>_A^2<\frac{x}{a}>_A + \frac{<\mu>_A^2}{4<a>_H^2}\int_0^1 a - \frac{<\mu>_A^2}{<a>_H}<\frac{x}{a}>_A\int_0^1 a + \frac{<\mu>_A^2}{<a>_H} \\
& -<\mu>_A^2<\frac{x}{a}>_A + <\mu>_A^2<\frac{x}{a}>_A^2\int_0^1 a - 2<\mu>_A^2<\frac{x}{a}>_A \\
& +2<\mu>_A^2<a>_H<\frac{x}{a}>_A^2 + \frac{<\mu>_A^2}{<a>_H} - 2<\mu>_A^2<\frac{x}{a}>_A \\
& +<\mu>_A^2<a>_H<\frac{x}{a}>_A^2 .
\end{aligned}
$$

However

$$\begin{aligned}
\int_0^1 B(x)dx &= \int_0^1\int_y^1 \frac{1}{a(y)}dxdy = \frac{1}{<a>_H} - <\frac{x}{a}>_A,\\
\int_0^1 xB &= \frac{1}{2<a>_H} - \frac{1}{2}\int_0^1 \frac{x^2}{a},\\
\int_0^1 xD &= \frac{1}{2}\left(<\mu>_A - \int_0^1 x^2\mu\right),
\end{aligned} \tag{4.4.23}$$

and

$$\begin{aligned}
\int_0^1 \frac{D^2}{a} &= \int_0^1 D^2(x)B'(x)dx\\
&= D^2(1)B(1) - 2\int_0^1 D(x)\mu(x)B(x) = \frac{<\mu>_A^2}{<a>_H} - 2\int_0^1 BD\mu.
\end{aligned} \tag{4.4.24}$$

Using the equations (4.4.20), (4.4.21), (4.4.23) and (4.4.24) leads to

$$\begin{aligned}
\int_0^1 a{\phi_2'}^2 = {} & <\mu>_A^2 \int_0^1 aB^2 + <a>_H\left(\int_0^1 B\mu\right)^2 + 2<\mu>_A\int_0^1 BD\\
& -2\frac{<\mu>_A^2}{<a>_H}\int_0^1 xaB - 2\int_0^1 BD\mu + 2<\mu>_A^2<\frac{x}{a}>_A\int_0^1 aB\\
& -\frac{<\mu>_A^2}{<a>_H}\int_0^1 aB - 2<a>_H\left(\int_0^1 B\mu\right)^2 + 2<\mu>_A\int_0^1 B\mu\\
& -2<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1 B\mu - 2<\mu>_A\int_0^1\frac{xD}{a} + 2<\mu>_A^2\int_0^1\frac{x^2}{a}\\
& +\frac{<\mu>_A^2}{<a>_H^2}\int_0^1 ax^2 + \frac{<\mu>_A}{<a>_H}\int_0^1 \mu x^2 - 2<\mu>_A<\frac{x}{a}>_A\int_0^1 \mu x\\
& +\frac{<\mu>_A}{<a>_H}\int_0^1 \mu x - \frac{11<\mu>_A^2}{6<a>_H} + 3<\mu>_A^2<\frac{x}{a}>_A\\
& -<\mu>_A^2<a>_H<\frac{x}{a}>_A^2 + \frac{<\mu>_A^2}{<a>_H^2}\int_0^1 ax - 2\frac{<\mu>_A^2}{<a>_H}<\frac{x}{a}>_A\int_0^1 xa\\
& +\frac{<\mu>_A^2}{4<a>_H^2}\int_0^1 a - \frac{<\mu>_A^2}{<a>_H}<\frac{x}{a}>_A\int_0^1 a + <\mu>_A^2<\frac{x}{a}>_A^2\int_0^1 a.
\end{aligned}$$

Putting the above result into $\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0)$, one gets

$$\begin{aligned}\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0) = & -<a>_H\left(\int_0^1 B\mu\right)^2 + 2<\mu>_A^2<a>_H\int_0^1 B^2\\ & -<\mu>_A<a>_H\int_0^1 B^2\mu + 2<\mu>_A\int_0^1 BD - 2\int_0^1 BD\mu\\ & +2<\mu>_A\int_0^1 xB\mu + 3<\mu>_A\int_0^1 B\mu\\ & -4<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1 B\mu - 2<\mu>_A\int_0^1\frac{xD}{a}\\ & +4<\mu>_A^2\int_0^1\frac{x^2}{a} - \frac{47<\mu>_A^2}{12<a>_H} + 6<\mu>_A^2<\frac{x}{a}>_A\\ & -4<\mu>_A^2<a>_H<\frac{x}{a}>_A^2.\end{aligned}$$

Since

$$\int_0^1 xB\mu = \frac{<\mu>_A}{<a>_H} - \int_0^1\frac{xD}{a} - \int_0^1 DB,$$

and using (4.4.24), it follows that

$$\begin{aligned}\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0) = & \int_0^1\frac{D^2}{a} - 4<\mu>_A\int_0^1\frac{xD}{a} + 4<\mu>_A^2\int_0^1\frac{x^2}{a}\\ & -<a>_H\left(\int_0^1 B\mu\right)^2 + 2<\mu>_A^2<a>_H\int_0^1 B^2\\ & -<\mu>_A<a>_H\int_0^1 B^2\mu - \frac{35<\mu>_A^2}{12<a>_H} + 3<\mu>_A\int_0^1 B\mu\\ & -4<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1 B\mu + 6<\mu>_A^2<\frac{x}{a}>_A\\ & -4<\mu>_A^2<a>_H<\frac{x}{a}>_A^2\end{aligned}$$

Furthermore,

$$\int_0^1 B^2\mu = B^2(1)D(1) - 2\int_0^1\frac{BD}{a} = \frac{<\mu>_A}{<a>_H^2} - 2\int_0^1\frac{BD}{a}. \quad (4.4.25)$$

Using (4.4.21) and (4.4.25), one gets that

$$\begin{aligned}
\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0) \;=\;& \int_0^1 \frac{(D-<\mu>_A\, x)^2}{a} - 2<\mu>_A\int_0^1\frac{xD}{a} + 3<\mu>_A^2\int_0^1\frac{x^2}{a}\\
&-<a>_H\left(\frac{<\mu>_A^2}{<a>_H^2} - 2\frac{<\mu>_A}{<a>_H}\int_0^1\frac{D}{a} + \left(\int_0^1\frac{D}{a}\right)^2\right)\\
&+2<\mu>_A^2<a>_H\int_0^1 B^2 - \frac{<\mu>_A^2}{<a>_H} + 2<\mu>_A<a>_H\int_0^1\frac{BD}{a}\\
&+3\frac{<\mu>_A^2}{<a>_H} - 3<\mu>_A\int_0^1\frac{D}{a} - 4<\mu>_A^2<\frac{x}{a}>_A\\
&+4<\mu>_A<a>_H<\frac{x}{a}>_A\int_0^1\frac{D}{a} - \frac{35<\mu>_A^2}{12<a>_H}\\
&+6<\mu>_A^2<\frac{x}{a}>_A - 4<\mu>_A^2<a>_H<\frac{x}{a}>_A^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{24}\times\frac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*,0) \;=\;& \int_0^1 \frac{(D-<\mu>_A\, x)^2}{a} - <a>_H\left(\int_0^1\frac{D}{a} - <\mu>_A<\frac{x}{a}>_A\right)^2\\
&+<\mu>_A\Bigg[<\mu>_A<a>_H\int_0^1 B^2 - <a>_H\int_0^1 B^2\mu\\
&+<\mu>_A<a>_H\int_0^1 B^2 + 2<a>_H<\frac{x}{a}>_A\int_0^1\frac{D}{a}\\
&-2\int_0^1\frac{xD}{a} - \int_0^1\frac{D}{a} + 3<\mu>_A\int_0^1\frac{x^2}{a} - 3<\mu>_A<a>_H<\frac{x}{a}>_A^2\\
&+2<\mu>_A<\frac{x}{a}>_A - \frac{11<\mu>_A}{12<a>_H}\Bigg]
\end{aligned}$$

$$\begin{aligned}
=\;& \int_0^1 \frac{(D-<\mu>_A\, x)^2}{a} - <a>_H\left(\int_0^1\frac{D}{a} - <\mu>_A<\frac{x}{a}>_A\right)^2\\
&+<\mu>_A\Bigg[<\mu>_A<a>_H\int_0^1\left(B - \frac{x}{<a>_H}\right)^2 - \frac{<\mu>_A}{4<a>_H}\\
&+<a>_H\int_0^1 B^2(<\mu>_A - \mu) + 2<a>_H<\frac{x}{a}>_A\int_0^1\frac{D}{a}\\
&-3<\mu>_A<a>_H<\frac{x}{a}>_A^2 - 2\int_0^1\frac{xD}{a} - \int_0^1\frac{D}{a}\\
&+2<\mu>_A<\frac{x}{a}>_A + 2<\mu>_A\int_0^1\frac{x^2}{a}\Bigg]
\end{aligned}$$

$$\begin{aligned}
= & \int_0^1 \frac{(D- <\mu>_A\ x)^2}{a} - < a >_H \left(\int_0^1 \frac{D}{a} - <\mu>_A < \frac{x}{a} >_A\right)^2 \\
& + <\mu>_A^2 < a >_H \left[\int_0^1 \left(B - \frac{x}{< a >_H}\right)^2 - \left(\frac{1}{2< a >_H} - <\frac{x}{a}>_A\right)^2\right] \\
& + <\mu>_A^2 <\frac{x}{a}>_A - 2 <\mu>_A^2 < a >_H <\frac{x}{a}>_A^2 + 2 <\mu>_A^2 \int_0^1 \frac{x^2}{a} \\
& + < a >_H <\mu>_A \int_0^1 B^2(<\mu>_A - \mu) \\
& + 2 <\mu>_A < a >_H <\frac{x}{a}>_A \int_0^1 \frac{D}{a} - 2 <\mu>_A \int_0^1 \frac{xD}{a} \\
& - <\mu>_A \int_0^1 \frac{D}{a}.
\end{aligned}$$

Moreover,

$$\int_0^1 B - \int_0^1 \frac{x}{< a >_H} = \frac{1}{2< a >_H} - <\frac{x}{a}>_A .$$

Thus,

$$\begin{aligned}
\frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) = & \int_0^1 \frac{(D- <\mu>_A\ x)^2}{a} - < a >_H \left(\int_0^1 \frac{D}{a} - <\mu>_A < \frac{x}{a} >_A\right)^2 \\
& + <\mu>_A^2 < a >_H \left[\int_0^1 \left(B - \frac{x}{< a >_H}\right)^2 - \left(\int_0^1 B - \int_0^1 \frac{x}{< a >_H}\right)^2\right] \\
& + C,
\end{aligned}$$

where

$$\begin{aligned}
C = & < a >_H <\mu>_A \int_0^1 B^2(<\mu>_A - \mu) + 2 <\mu>_A < a >_H <\frac{x}{a}>_A \int_0^1 \frac{D}{a} \\
& - 2 <\mu>_A \int_0^1 \frac{xD}{a} - <\mu>_A \int_0^1 \frac{D}{a} + <\mu>_A^2 <\frac{x}{a}>_A - 2 <\mu>_A^2 < a >_H <\frac{x}{a}>_A^2 \\
& + 2 <\mu>_A^2 <\frac{x^2}{a}>_A .
\end{aligned}$$

Since $B'(x) = \dfrac{1}{a(x)}$ over $\mathbb{R}$, it follows that

$$\int_0^1 \frac{Bx}{a} = B^2(1) - \int_0^1 BB'x - \int_0^1 B^2.$$

Consequently,

$$\int_0^1 B^2 = \frac{1}{<a>_H^2} - 2\int_0^1 \frac{Bx}{a}. \tag{4.4.26}$$

Together with (4.4.25), it follows that

$$\int_0^1 B^2(<\mu>_A - \mu) = 2\int_0^1 \frac{B(D- <\mu>_A x)}{a}.$$

Putting the above result into the expression of C, one then obtains

$$\begin{aligned} C &= 2<\mu>_A<a>_H<\frac{x}{a}>_A \int_0^1 \frac{D- <\mu>_A x}{a} \\ &\quad - <\mu>_A \int_0^1 \frac{D- <\mu>_A x}{a} - 2<\mu>_A \int_0^1 \frac{(D- <\mu>_A x)x}{a} \\ &\quad +2<\mu>_A<a>_H \int_0^1 \frac{B(D- <\mu>_A x)}{a} \\ &= 2<\mu>_A<a>_H \int_0^1 \left(\frac{D- <\mu>_A x}{a}\right)\left(B - \frac{x}{<a>_H}\right) \\ &\quad +2<\mu>_A<a>_H \int_0^1 \frac{D- <\mu>_A x}{a} \int_0^1 \left(\frac{x}{a} - \frac{x}{<a>_H}\right) \\ &= 2<\mu>_A<a>_H \int_0^1 \left(\frac{D- <\mu>_A x}{a}\right)\left(B - \frac{x}{<a>_H}\right) \\ &\quad -2<\mu>_A<a>_H \int_0^1 \frac{D- <\mu>_A x}{a} \int_0^1 \left(B - \frac{x}{<a>_H}\right). \end{aligned} \tag{4.4.27}$$

For all $x \in \mathbb{R}$, let

$$E(x) = D(x)- <\mu>_A x, \quad \text{and} \quad F(x) = B(x) - \frac{x}{<a>_H}.$$

Thus,

$$C = 2<\mu>_A<a>_H \int_0^1 \frac{EF}{a} - 2<\mu>_A<a>_H \int_0^1 \frac{E}{a}\int_0^1 F,$$

whence,

$$\begin{aligned} \frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) &= \int_0^1 \frac{E^2}{a} - <a>_H \left(\int_0^1 \frac{E}{a}\right)^2 \\ &\quad + <\mu>_A^2<a>_H \left[\int_0^1 F^2 - \left(\int_0^1 F\right)^2\right] \\ &\quad +2<\mu>_A<a>_H \int_0^1 \frac{EF}{a} - 2<\mu>_A<a>_H \int_0^1 \frac{E}{a}\int_0^1 F. \end{aligned}$$

Since

$$\int_0^1 F = <a>_H \int_0^1 \frac{F}{a} \quad \text{and} \quad \int_0^1 F^2 = <a>_H \int_0^1 \frac{F^2}{a},$$

one concludes that

$$\begin{aligned}
\frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) &= \int_0^1 \frac{E^2}{a} + <\mu>_A^2 <a>_H^2 \int_0^1 \frac{F^2}{a} + 2 <\mu>_A <a>_H \int_0^1 \frac{EF}{a} \\
&\quad - <a>_H \left(\int_0^1 \frac{E}{a} \right)^2 - <\mu>_A^2 <a>_H^3 \left(\int_0^1 \frac{F}{a} \right)^2 \\
&\quad -2 <\mu>_A <a>_H^2 \int_0^1 \frac{E}{a} \int_0^1 F \\
&= \int_0^1 \frac{(E + <\mu>_A <a>_H F)^2}{a} \\
&\quad - <a>_H \left(\int_0^1 \frac{E}{a} + <\mu>_A <a>_H \int_0^1 \frac{F}{a} \right)^2.
\end{aligned}$$

For all $x \in \mathbb{R}$, denote

$$\begin{aligned}
A(x) &= E(x) + <\mu>_A <a>_H F(x) \\
&= \int_0^x \mu(y) dy + <\mu>_A <a>_H \int_0^x \frac{1}{a(y)} dy - 2 <\mu>_A x.
\end{aligned}$$

It follows that

$$\frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) = \int_0^1 \frac{A(x)^2}{a(x)} dx - <a>_H \left(\int_0^1 \frac{A(x)}{a(x)} dx \right)^2.$$

From (4.4.9), one concludes that

$$\frac{d^2 c_L^*}{dL^2} \to \gamma := 2\sqrt{<a>_H <\mu>_A^{-1}} \times \left[\int_0^1 \frac{A(x)^2}{a(x)} dx - <a>_H \left(\int_0^1 \frac{A(x)}{a(x)} dx \right)^2 \right] \text{ as } L \to 0^+.$$

Cauchy-Schwarz inequality and the fact $a(x) > 0$ in $\mathbb{R}$ yield that $\gamma \geq 0$. Furthermore, $\gamma = 0$ if and only if A is constant. But since $A(0) = 0$, the condition $\gamma = 0$ is equivalent to $A'(x) = 0$ for all x, which means that

$$\frac{\mu(x)}{<\mu>_A} + \frac{<a>_H}{a(x)} = 2 \text{ for all } x \in \mathbb{R}.$$

In particular, if μ is constant and a is not constant (resp. if a is constant and μ is not constant), then this condition is not satisfied, whence $\lim_{L \to 0^+} \frac{d^2 c_L^*}{dL^2} > 0$ in this case. That completes the proofs of Theorem 4.2.3 and Corollary 4.2.4. □

Remark 4.4.2 In the case when $<\mu>_A=0$ and $\mu \not\equiv 0$, then $\rho_{1,L}<0$ for each $L>0$, and the minimal speed c_L^* of pulsating traveling fronts is well-defined and it is still positive. From the arguments developed in this section and in the previous one, one can check that, in this case,

$$c_L^* \to 0^+, \quad \lambda_L^* \to 0^+, \quad \frac{d\lambda_L^*}{dL} \to \sqrt{\frac{\beta}{<a>_H}} > 0 \text{ and } \frac{dc_L^*}{dL} \to 2\sqrt{\beta <a>_H} > 0 \text{ as } L \to 0^+,$$

where

$$\beta = \int_0^1 \frac{A(x)^2}{a(x)}dx - \ <a>_H \left(\int_0^1 \frac{A(x)}{a(x)}dx\right)^2 > 0$$

and $A(x) = \displaystyle\int_0^x \mu(y)dy$. Therefore, the speeds c_L^* are increasing in a right neighbourhood of $L=0$ but, in this case, the variation is of the first order. Notice that the formula $\lim_{L\to 0^+} \frac{dc_L^*}{dL} = 2\sqrt{\beta <a>_H}$ is coherent with the numerical calculations done by Kinezaki, Kawasaki and Shigesada in [21] (see Figure 3b with $<\mu>_A=0$, that is $A=0$ under the notations of [21]).

4.5 Proof of Theorem 4.2.6

As in the proofs of the previous theorems, we use the following formula for the minimal speed:

$$c_z^* = \min_{\lambda>0} \frac{k_z(\lambda)}{\lambda} = \frac{k_z(\lambda_z^*)}{\lambda_z^*}, \tag{4.5.1}$$

where $k_z(\lambda)$ is defined as the unique real number such that there exists a positive L_0-periodic function ψ satisfying:

$$\psi'' + 2\lambda\ \psi' + \lambda^2\psi + \mu_z(x)\psi = k_z(\lambda)\psi \text{ in } (0, L_0). \tag{4.5.2}$$

Setting $\varphi(x) = e^{\lambda x}\psi(x)$, the above equation and periodicity conditions become equivalent to:

$$\begin{cases} \varphi'' + \mu_z(x)\varphi = k_z(\lambda)\varphi \text{ in } (0, L_0), \\ \varphi(L_0) = e^{\lambda L_0}\varphi(0), \\ \varphi'(L_0) = e^{\lambda L_0}\varphi'(0), \end{cases} \tag{4.5.3}$$

which therefore admits, for every positive λ, a unique solution $(\varphi, k_z(\lambda))$ with $\varphi > 0$ satisfying the normalisation condition $\varphi(0) = 1$.

Let $\lambda > 0$ be fixed. System (4.5.3), together with the normalization condition $\varphi(0) = 1$, is equivalent to:

$$\begin{cases} \varphi'' &= (k_z(\lambda) - m)\varphi \text{ on } [0, l/2), \\ \varphi'' &= k_z(\lambda)\varphi \text{ on } [l/2, l/2 + z), \\ \varphi'' &= (k_z(\lambda) - m)\varphi \text{ on } [l/2 + z, l + z), \\ \varphi'' &= k_z(\lambda)\varphi \text{ on } [l + z, L_0), \\ \varphi(0) &= 1,\ \varphi(L_0) = e^{\lambda L_0}\varphi(0),\ \varphi'(L_0) = e^{\lambda L_0}\varphi'(0). \end{cases} \tag{4.5.4}$$

For each $z \in [0, L_0 - l]$, let λ_z^* be defined by the formula (4.5.1). We have the following lemma:

Lemma 4.5.1 *Assume that $l > 3L_0/4$. Then, for all $z \in [0, L_0 - l]$, we have $k_z(\lambda_z^*) > m$.*

Proof of Lemma 4.5.1. Let us divide equation (4.5.2) by ψ and integrate by parts over $[0, L_0]$. Using the L_0-periodicity of ψ, we obtain:

$$\int_0^{L_0} \frac{|\psi'|^2}{\psi^2} + L_0\lambda^2 + \int_0^{L_0} \mu_z(x)dx = L_0 k_z(\lambda).$$

Thus,

$$k_z(\lambda) \geq \lambda^2 + \frac{1}{L_0}\int_0^{L_0} \mu_z(x)dx = \lambda^2 + m\frac{l}{L_0}. \tag{4.5.5}$$

From (4.5.1) and (4.5.5) we get:

$$(\lambda_z^*)^2 + m\frac{l}{L_0} \leq k_z(\lambda_z^*) \leq 2\lambda_z^*\sqrt{m}.$$

Thus, $(\lambda_z^*)^2 - 2\lambda_z^*\sqrt{m} + ml/L_0 \leq 0$, which implies that

$$\lambda_z^* \geq \sqrt{m} - \sqrt{m - ml/L_0}.$$

Using (4.5.5), we finally get

$$k_z(\lambda_z^*) \geq 2m(1 - \sqrt{1 - l/L_0}) > m,$$

as soon as $l > 3L_0/4$. □

We now turn to the proof of Theorem 4.2.6 and we assume that $l \in (3L_0/4, L_0)$. Using the fact that $\varphi \in C^1(\mathbb{R})$, a straightforward but lengthy computation shows that, whenever $k_z(\lambda) > m$, system (4.5.4) is equivalent to

$$\frac{F(z, \lambda, k_z(\lambda))}{G(z, k_z(\lambda))} = 0,$$

where F and G are two functions, defined respectively in $[0, L_0 - l] \times (0, +\infty) \times [m, +\infty)$ and $[0, L_0 - l] \times [m, +\infty)$ by:

$$\begin{aligned} F(z, \lambda, s) = \quad & 4(2s - m)\sqrt{s}\sqrt{s-m}\sinh(l\sqrt{s-m})\sinh(\alpha\sqrt{s}) \\ & + m^2\cosh(\beta\sqrt{s})(1 - \cosh(l\sqrt{s-m})) \\ & + 8(s^2 - ms)[\cosh(l\sqrt{s-m})\cosh(\alpha\sqrt{s}) - \cosh(\lambda L_0)] \\ & + m^2\cosh(\alpha\sqrt{s})(\cosh(l\sqrt{s-m}) - 1), \end{aligned} \tag{4.5.6}$$

and

$$\begin{aligned} G(z, s) = \quad & m\sqrt{s}\cosh(l\sqrt{s-m})\,[4\sinh(\alpha\sqrt{s})(s/m - 1) \\ & + (\sinh(\alpha\sqrt{s}) - \sinh(\beta\sqrt{s}))\left(1 - 1/\cosh(l\sqrt{s-m})\right)] \\ & + m\sqrt{s-m}\sinh(l\sqrt{s-m})\cosh(\alpha\sqrt{s})\left[\tfrac{4s}{m} - 1 + \tfrac{\cosh(\beta\sqrt{s})}{\cosh(\alpha\sqrt{s})}\right], \end{aligned} \tag{4.5.7}$$

with $\alpha := L_0 - l$ and $\beta := L_0 - l - 2z$.

Each factor in the expression (4.5.7) is positive, as soon as $s > m$, for $z \in [0, L_0 - l]$. Thus, whenever $k_z(\lambda) > m$, system (4.5.4) is equivalent to the simpler equation

$$F(z, \lambda, k_z(\lambda)) = 0. \tag{4.5.8}$$

Furthermore, from Krein-Rutman theory, since the eigenfunction ψ in (4.5.2) is positive, $k_z(\lambda)$ is the largest real eigenvalue of the operator $\psi \mapsto \psi'' + 2\lambda\,\psi' + \lambda^2\psi + \mu_z(x)\psi$. This result, implies that, for each $z \in [0, L_0 - l]$, and each $\lambda > 0$, $k_z(\lambda)$ is the largest real root of equation (4.5.8), as soon as $k_z(\lambda) > m$.

From equation (4.5.6), we easily see that

$$\lim_{s \to +\infty} F(z, \lambda, s) = +\infty, \tag{4.5.9}$$

for all $z \in [0, L_0 - l]$ and $\lambda > 0$. Moreover, differentiating (4.5.6) with respect to z, we obtain

$$\frac{\partial F}{\partial z}(z, \lambda, s) = 2m^2\sqrt{s}\sinh(\sqrt{s}(L_0 - l - 2z))\left[\cosh(l\sqrt{s-m}) - 1\right].$$

Thus, for all $s > m$, and $\lambda > 0$,

$$\frac{\partial F}{\partial z}(z, \lambda, s) > 0 \text{ for } z \in [0, (L_0 - l)/2), \tag{4.5.10}$$

and

$$\frac{\partial F}{\partial z}(z, \lambda, s) < 0 \text{ for } z \in ((L_0 - l)/2, L_0 - l].$$

Now, take $z_1 < z_2$ in $[0, (L_0 - l)/2]$, and assume that $c^*_{z_1} \leq c^*_{z_2}$. It follows from formula (4.5.1) that $k_{z_2}(\lambda) \geq c^*_{z_2}\lambda$, for all $\lambda > 0$. In particular,

$$k_{z_2}(\lambda^*_{z_1}) \geq c^*_{z_2}\lambda^*_{z_1} \geq c^*_{z_1}\lambda^*_{z_1} = k_{z_1}(\lambda^*_{z_1}). \tag{4.5.11}$$

From Lemma 4.5.1, we know that $k_{z_1}(\lambda^*_{z_1}) > m$. Thus, (4.5.11) implies $k_{z_2}(\lambda^*_{z_1}) > m$. From the above discussion, $k_{z_2}(\lambda^*_{z_1})$ is therefore the largest real root of the equation $F(z_2, \lambda^*_{z_1}, k_{z_2}(\lambda^*_{z_1})) = 0$, and, similarly, $k_{z_1}(\lambda^*_{z_1})$ is the largest real root of $F(z_1, \lambda^*_{z_1}, k_{z_1}(\lambda^*_{z_1})) = 0$. Using (4.5.9) and (4.5.10), and since $0 \leq z_1 < z_2 \leq (L_0 - l)/2$, we obtain $k_{z_2}(\lambda^*_{z_1}) < k_{z_1}(\lambda^*_{z_1})$, which contradicts (4.5.11). Therefore, c^*_z is a decreasing function of z in $[0, (L_0 - l)/2]$. Similar arguments imply that c^*_z is an increasing function of z in $[(L_0 - l)/2, L_0 - l]$. This concludes the proof of Theorem 4.2.6. □

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