Annali di Matematica (2010) 189:47–66
DOI 10.1007/s10231-009-0100-2

Min–max formulæ for the speeds of pulsating travelling fronts in periodic excitable media

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Received: 27 May 2008 / Accepted: 25 February 2009 / Published online: 26 March 2009

Abstract This paper is concerned with some nonlinear propagation phenomena for reaction–advection–diffusion equations in a periodic framework. It deals with travelling wave solutions of the equation

$$u_t = \nabla \cdot (A(z)\nabla u) \; + q(z) \cdot \nabla u + \; f(z, u), \qquad t \in \mathbb{R}, \quad z \in \Omega,$$

propagating with a speed c. In the case of a "combustion" nonlinearity, the speed c exists and it is unique, while the front u is unique up to a translation in t. We give a min–max and a max–min formula for this speed c. On the other hand, in the case of a "ZFK" or a "KPP" nonlinearity, there exists a minimal speed of propagation c^*. In this situation, we give a min–max formula for c^*. Finally, we apply this min–max formula to prove a variational formula involving eigenvalue problems for the minimal speed c^* in the "KPP" case.

Keywords Reaction–diffusion equations · Travelling fronts · Periodic media · Min–max formulas · Speeds of propagation

Mathematics Subject Classification (2000) Primary: 35K55 · 35K57 · 35B10 · 35B50 · 35J60; Secondary: 35K20 · 35B30

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1 Introduction and main results

1.1 A description of the periodic framework

The goal of this paper is to give some variational formulæ for the speeds of pulsating travelling fronts corresponding to reaction–diffusion–advection equations set in a heterogenous periodic framework. In fact, many works, such as Hamel [6], Heinze et al. [9], and Volpert et al. [16] treated this problem in simplified situations and under more strict assumptions. In this paper, we treat the problem in the most general periodic framework. We are concerned with equations of the type

$$\begin{cases} u_t = \nabla\cdot(A(z)\nabla u) + q(z)\cdot\nabla u + f(z,u), & t\in\mathbb{R},\ z\in\Omega,\\ \nu\cdot A\ \nabla u(t,z) = 0, & t\in\mathbb{R},\ z\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\nu(z)$ is the unit outward normal on $\partial\Omega$ at the point z. In this context, let us detail the mathematical description of the heterogeneous setting.

Concerning the domain, let $N\geq 1$ be the space dimension, and let d be an integer so that $1\leq d\leq N$. For an element $z=(x_1,x_2,\ldots,x_d,x_{d+1},\ldots,x_N)\in\mathbb{R}^N$, we denote by $x=(x_1,x_2,\ldots,x_d)$ and by $y=(x_{d+1},\ldots,x_N)$ the two tuples so that $z=(x,y)$. Let $L_1,\ldots,L_d$ be d positive real numbers, and let Ω be a C^3 non empty connected open subset of $\mathbb{R}^N$ satisfying

$$\begin{cases} \exists R\geq 0;\ \forall (x,y)\in\Omega,\ |y|\leq R,\\ \forall\,(k_1,\ldots,k_d)\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z},\quad \Omega=\Omega+\sum_{k=1}^d k_i e_i, \end{cases} \tag{1.2}$$

where $(e_i)_{1\leq i\leq N}$ is the canonical basis of $\mathbb{R}^N$.

As $d\geq 1$, one notes that the set Ω satisfying (1.2) is unbounded. We have many archetypes of such a domain. The case of the whole space $\mathbb{R}^N$ corresponds to $d=N$, where $L_1,\ldots,L_N$ are any positive numbers. The case of the whole space $\mathbb{R}^N$ with a periodic array of holes can also be considered. The case $d=1$ corresponds to domains which have only one unbounded dimension, namely infinite cylinders which may be straight or have oscillating periodic boundaries, and which may or may not have periodic perforations. The case $2\leq d\leq N-1$ corresponds to infinite slabs.

In this periodic situation, we give the following definitions:

Definition 1.1 (*Periodicity cell*) The set

$$C=\{(x,y)\in\Omega;\ x_1\in(0,L_1),\ldots,x_d\in(0,L_d)\}$$

is called the periodicity cell of Ω.

Definition 1.2 (*L-periodic fields*) A field $w:\Omega\to\mathbb{R}^N$ is said to be L-periodic with respect to x if

$$w(x_1+k_1,\ldots,x_d+k_d,y)=w(x_1,\ldots,x_d,y)$$

almost everywhere in Ω, and for all $k=(k_1,\ldots,k_d)\in\prod_{i=1}^d L_i\mathbb{Z}$.

Let us now detail the assumptions concerning the coefficients in (1.1). First, the diffusion matrix $A(x,y)=(A_{ij}(x,y))_{1\leq i,j\leq N}$ is a symmetric $C^{2,\delta}(\overline{\Omega})$ (with $\delta>0$) matrix field satisfying

$$\begin{cases} A \text{ is } L\text{-periodic with respect to } x, \\ \exists\, 0 < \alpha_1 \leq \alpha_2; \forall (x,y) \in \Omega, \forall \xi \in \mathbb{R}^N, \\ \alpha_1 |\xi|^2 \leq \sum_{1 \leq i,j \leq N} A_{ij}(x,y)\xi_i \xi_j \leq \alpha_2 |\xi|^2. \end{cases} \tag{1.3}$$

The underlying advection $q(x,y) = (q_1(x,y), \dots, q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) vector field satisfying:

$$\begin{cases} q \text{ is } L\text{-periodic with respect to } x, \\ \nabla \cdot q = 0 \quad \text{in } \overline{\Omega}, \\ q \cdot \nu = 0 \quad \text{on } \partial\Omega, \\ \forall 1 \leq i \leq d, \quad \int_C q_i \, \mathrm{d}x \, \mathrm{d}y = 0. \end{cases} \tag{1.4}$$

Lastly, let $f = f(x,y,u)$ be a function defined in $\overline{\Omega} \times \mathbb{R}$ such that

$$\begin{cases} f \text{ is globally Lipschitz-continuous in } \overline{\Omega} \times \mathbb{R}, \\ \forall (x,y) \in \overline{\Omega}, \forall s \in (-\infty, 0] \cup [1, +\infty), f(s,x,y) = 0, \\ \exists \rho \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall 1 - \rho \leq s \leq s' \leq 1, \\ f(x,y,s) \geq f(x,y,s'). \end{cases} \tag{1.5}$$

One assumes that

$$f \text{ is } L\text{-periodic with respect to } x. \tag{1.6}$$

Moreover, the function f is assumed to be of one of the following two types: either

$$\begin{cases} \exists\, \theta \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall s \in [0,\theta],\ f(x,y,s) = 0, \\ \forall s \in (\theta, 1),\ \exists\, (x,y) \in \overline{\Omega} \ \text{ such that } f(x,y,s) > 0, \end{cases} \tag{1.7}$$

or

$$\begin{cases} \exists \delta > 0,\ \text{the restriction of } f \text{ to } \overline{\Omega} \times [0,1] \text{ is of class } C^{1,\delta}, \\ \forall s \in (0,1), \exists\, (x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0. \end{cases} \tag{1.8}$$

Definition 1.3 A nonlinearity f satisfying (1.5), (1.6) and (1.7) is called a "combustion" nonlinearity. The value θ is called the ignition temperature.

A nonlinearity f satisfying (1.5), (1.6), and (1.8) is called a ZFK (for Zeldovich–Frank–Kamenetskii) nonlinearity.

If f is a "ZFK" nonlinearity that satisfies

$$f'_u(x,y,0) = \lim_{u \to 0^+} f(x,y,u)/u > 0, \tag{1.9}$$

with the additional assumption

$$\forall (x,y,s) \in \overline{\Omega} \times (0,1), \quad 0 < f(x,y,s) \leq f'_u(x,y,0) \times s, \tag{1.10}$$

then f is called a KPP (for Kolmogorov, Petrovsky, and Piskunov, see [11]) nonlinearity.

The simplest examples of "combustion" and "ZFK" nonlinearities are when $f(x,y,u) = f(u)$ where: either

$$\begin{cases} f \text{ is Lipschitz-continuous in } \mathbb{R}, \\ \exists \theta \in (0,1),\ f(s) = 0 \text{ for all } s \in (-\infty, \theta] \cup [1, +\infty), \\ f(s) > 0 \text{ for all } s \in (\theta, 1), \\ \exists\ \rho \in (0, 1-\theta), \quad f \text{ is non-increasing on } [1-\rho, 1], \end{cases} \tag{1.11}$$

or

$$\begin{cases} f \text{ is defined on } \mathbb{R},\ f \equiv 0 \text{ in } \mathbb{R}\backslash(0,1), \\ \exists\ \delta > 0,\ \text{the restriction of } f \text{ on the interval } [0,1] \text{ is } C^{1,\delta}([0,1]), \\ f(0) = f(1) = 0,\ \text{and } f(s)\ > 0\ \text{ for all } s \in (0,1), \\ \exists\ \rho > 0, \quad f \text{ is non-increasing on } [1-\rho, 1]. \end{cases} \tag{1.12}$$

If $f = f(u)$ satisfies (1.11), then it is a homogeneous "combustion" nonlinearity. On the other hand, a nonlinearity $f = f(u)$ that satisfies (1.12) is homogeneous of the "ZFK" type. Moreover, a KPP homogeneous nonlinearity is a function $f = f(u)$ that satisfies (1.12) with the additional assumption

$$\forall\, s \in (0,1), \quad 0 < f(s) \le f'(0)\, s. \tag{1.13}$$

As typical examples of nonlinear heterogeneous sources satisfying (1.5) and (1.6) and either (1.7) or (1.8), one can consider the functions of the type

$$f(x, y, u) = h(x, y)\, g(u),$$

where h is a globally Lipschitz-continuous, positive, bounded, and L-periodic with respect to x function defined in $\overline{\Omega}$, and g is a function satisfying either (1.11) or (1.12).

Definition 1.4 (*Pulsating fronts and speed of propagation*) Let $e = (e^1, \ldots, e^d)$ be an arbitrarily given unit vector in $\mathbb{R}^d$. A function $u = u(t, x, y)$ is called a pulsating travelling front propagating in the direction of $-e$ with an effective speed $c \neq 0$, if u is a classical solution of:

$$\begin{cases} u_t = \nabla \cdot (A(x,y)\nabla u) + q(x,y) \cdot \nabla u + \ f(x,y,u),\ t \in \mathbb{R},\ (x,y) \in \Omega, \\ \nu \cdot A\ \nabla u(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega, \\ \forall\, k \in \prod_{i=1}^d L_i\mathbb{Z},\ \forall\,(t,x,y) \in \mathbb{R} \times \overline{\Omega},\ u\left(t + \frac{k\cdot e}{c}, x, y\right) = u(t, x+k, y), \\ \lim_{x\cdot e \to -\infty} u(t,x,y) = 0,\ \text{and}\ \lim_{x\cdot e\to +\infty} u(t,x,y) = 1, \\ 0 \le u \le 1, \end{cases} \tag{1.14}$$

where the above limits hold locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e .

Several works were concerned with pulsating travelling fronts in periodic media (see [1,2,10,12,14,15,18]).

In the general periodic framework, we recall two essential known results and then we move to our main results.

Theorem 1.5 (Berestycki and Hamel [1]) *Let Ω be a domain satisfying* (1.2), *let e be any unit vector of $\mathbb{R}^d$ and let f be a nonlinearity satisfying* (1.5) *and* (1.6) *and* (1.7). *Assume, furthermore, that A and q satisfy* (1.3) *and* (1.4) *respectively. Then, there exists a classical solution (c, u) of* (1.14). *Moreover, the speed c is positive and unique while the function $u = u(t, x, y)$ is increasing in t and it is unique up to a translation. Precisely, if (c^1, u^1) and*

(c^2, u^2) are two classical solutions of (1.14), *then $c^1 = c^2$ and there exists $h \in \mathbb{R}$ such that $u^1(t, x, y) = u^2(t + h, x, y)$ for all $(t, x, y) \in \mathbb{R} \times \overline{\Omega}$.*

In a periodic framework, Theorem 1.5 yields the existence of a pulsating travelling front in the case of a "combustion" nonlinearity with an ignition temperature θ. It implies, also, the uniqueness of the speed and of the profile of u. For "ZFK" nonlinearities, we have

Theorem 1.6 (Berestycki and Hamel [1]) *Let Ω be a domain satisfying* (1.2), *let e be any unit vector in $\mathbb{R}^d$ and let f be a nonlinearity satisfying* (1.5) *and* (1.6) *and* (1.8). *Assume, furthermore, that A and q satisfy* (1.3) *and* (1.4) *respectively. Then, there exists $c^*_{\Omega,A,q,f}(e) > 0$ such that the problem* (1.14) *has no solution (c, u) such that $u_t > 0$ in $\mathbb{R} \times \overline{\Omega}$ if $c < c^*_{\Omega,A,q,f}(e)$ while, for each $c \geq c^*_{\Omega,A,q,f}(e)$, it has a solution (c, u) such that u is increasing in t.*

In fact, the existence and the monotonicity of a solution $u^* = u^*(t, x, y)$ of (1.14) for $c = c^*_{\Omega,A,q,f}(e) > 0$ holds by approaching the "ZFK" nonlinearity f by a sequence of combustion nonlinearities $(f_\theta)_\theta$ such that $f_\theta \to f$ uniformly in $\mathbb{R} \times \overline{\Omega}$ as $\theta \searrow 0^+$ (see more details in step 2 of the proof of formula (1.17), Sect. 4). It follows, from Theorem 1.5, that for each $\theta > 0$, there exists a solution (c_θ, u_θ) of (1.14) with the nonlinearity f_θ such that u_θ is increasing with respect to t. From parabolic estimates, the functions u_θ, converge up to a subsequence, to a function u^* in $C^2_{loc}(\mathbb{R} \times \overline{\Omega})$ as $\theta \to 0^+$. Moreover, Lemmas 6.1 and 6.2 in [1] yield the existence of a constant $c^*(e) = c^*_{\Omega,A,q,f}(e) > 0$ such that $c_\theta \nearrow c^*(e)$ as $\theta \searrow 0$. Hence, the couple $(c^*(e), u^*)$ becomes a classical solution of (1.14) with the nonlinearity f and one gets that u^* is nondecreasing with respect to t as a limit of the increasing functions u_θ. Finally, one applies the strong parabolic maximum principle and Hopf lemma to get that w is positive in $\mathbb{R} \times \overline{\Omega}$. In other words, u^* is increasing in $t \in \mathbb{R}$. Actually, in the "ZFK" case, under the additional non-degeneracy assumption (1.9), it is known that any pulsating front with speed c is increasing in time and $c \geq c^*(e)$ (see [1]).

The value $c^*_{\Omega,A,q,f}(e)$ which appears in Theorem 1.6 is called the minimal speed of propagation of the pulsating travelling fronts propagating in the direction $-e$ (satisfying the reaction–advection–diffusion problem (1.14)).

We mention that the uniqueness of the pulsating travelling fronts, up to shifts in time, for each $c \geq c^*_{\Omega,A,q,f}(e)$, has been proved recently by Hamel and Roques [7] for "KPP" nonlinearities. On the other hand, a variational formula for the minimal speed of propagation $c^*_{\Omega,A,q,f}(e)$, in the case of a KPP nonlinearity, was proved in Berestycki et al. [2]. This formula involves eigenvalue problems and gives the value of the minimal speed in terms of the domain Ω and in terms of the coefficients A, q, and f appearing in problem (1.14). The asymptotic behaviors and the variations of the minimal speed of propagation, as a function of the diffusion, advection and reaction factors and as a function of the periodicity parameters, were widely studied in Berestycki et al. [3], El Smaily [4], El Smaily et al. [5], Heinze [8], Ryzhik and Zlatoš [13], and Zlatoš [20].

1.2 Main results

In the periodic framework, having (in Theorems 1.5 and 1.6) the existence results and some qualitative properties of the pulsating travelling fronts propagating in the direction of a fixed unit vector $-e \in \mathbb{R}^d$, we search a variational formula for the unique speed of propagation $c = c(e)$ whenever f is of the "combustion" type, and for the minimal speed $c^* = c^*_{\Omega,A,q,f}(e)$ whenever f is of the "ZFK" or the "KPP" type. We will answer the above investigations in the following theorem, but before this, we introduce the following

Notation 1.7 *For each function $\phi = \phi(s, x, y)$ in $C^{1,\delta}(\mathbb{R} \times \overline{\Omega})$ (for some $\delta \in [0, 1)$), let*

$$F[\phi] := \nabla_{x,y} \cdot (A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \nabla_{x,y} \cdot (A\tilde{e}\phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi) \text{ in } \mathcal{D}'(\mathbb{R} \times \Omega),$$

where $\tilde{e} = (e, 0, \ldots, 0) \in \mathbb{R}^N$ and e denotes a unit vector of $\mathbb{R}^d$.

The first main result deals with the "combustion" case.

Theorem 1.8 *Let e a unit vector of $\mathbb{R}^d$. Assume that Ω is a domain satisfying (1.2) and f is a nonlinear source satisfying (1.5) and (1.6). Assume furthermore that A and q satisfy (1.3) and (1.4) respectively. Consider the set of functions*

$$\begin{aligned} E = \big\{&\varphi = \varphi(s, x, y),\ \varphi \text{ is of class } C^{1,\mu}(\mathbb{R} \times \overline{\Omega}) \text{ for each } \mu \in [0, 1),\\ &F[\varphi] \in C(\mathbb{R} \times \overline{\Omega}),\ \varphi \text{ is } L\text{-periodic with respect to } x,\ \varphi_s > 0\\ &\text{in } \mathbb{R} \times \overline{\Omega},\ \varphi(-\infty, \cdot, \cdot) = 0,\ \varphi(+\infty, \cdot, \cdot) = 1 \text{ uniformly in } \overline{\Omega}, \text{ and}\\ &\nu \cdot A(\nabla_{x,y}\varphi + \tilde{e}\varphi_s) = 0 \text{ on } \mathbb{R} \times \partial\Omega\big\}. \end{aligned}$$

For each $\varphi \in E$, we define the function $R\varphi \in C(\mathbb{R} \times \overline{\Omega})$ as, for all $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$,

$$R\,\varphi(s, x, y) = \frac{F[\varphi](s, x, y) + q \cdot \nabla_{x,y}\varphi(s, x, y) + f(x, y, \varphi)}{\partial_s\varphi(s, x, y)} + q(x, y) \cdot \tilde{e}.$$

If f is a nonlinearity of "combustion" type satisfying (1.7), then the unique speed $c(e)$ that corresponds to problem (1.14) is given by

$$c(e) = \min_{\varphi \in E}\ \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y), \tag{1.15}$$

$$c(e) = \max_{\varphi \in E}\ \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y). \tag{1.16}$$

Furthermore, the min *in (1.15) and the* max *in (1.16) are attained by, and only by, the function $\phi(s, x, y) = u(\frac{s - x \cdot e}{c(e)}, x, y)$ and its shifts $\phi(s + \tau, x, y)$ for any $\tau \in \mathbb{R}$, where u is the solution of (1.14) with a speed $c(e)$ (whose existence and uniqueness up to a translation in t follow from Theorem 1.5).*

The second result is concerned with "ZFK" nonlinearities.

Theorem 1.9 *Under the same notations of Theorem 1.8, if f is a nonlinearity of "ZFK" type satisfying (1.5), (1.6) and (1.8), then the minimal speed $c^*_{\Omega,A,q,f}(e)$ is given by*

$$c^*_{\Omega,A,q,f}(e) = \min_{\varphi \in E}\ \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y). \tag{1.17}$$

Furthermore, the min *is attained by the function $\phi^*(s, x, y) = u^*(\frac{s - x \cdot e}{c^*(e)}, x, y)$ and its shifts $\phi^*(s + \tau, x, y)$ for any $\tau \in \mathbb{R}$, where u^* is any solution of (1.14) propagating with the speed $c^*(e) = c^*_{\Omega,A,q,f}(e)$.*

In particular, Theorem 1.9 yields that formula (1.17) holds in the "KPP" case (1.10) as well.

Remark 1.10 In Theorem 1.8, the min and the max are attained by, and only by, the pulsating front $\phi(s, x, y)$ and its shifts $\phi(s + \tau, x, y)$ for all $\tau \in \mathbb{R}$. In Theorem 1.9, the min is achieved by the front $\phi^*(s, x, y)$ with the speed $c^*(e)$ and all its shifts $\phi^*(s + \tau, x, y)$. Actually, if the pulsating front ϕ^* is unique up to shift, then ϕ^* and its shifts are the unique minimizers in formula (1.17). The uniqueness is known in the "KPP" case (see Hamel and Roques [7]), but it is still open in the general "ZFK" case.

We mention that a max–min formula of the type (1.16) can not hold for the minimal speed $c^*(e)$ in the "ZFK" or the "KPP" case. A simple justification is given in Sect. 2.

The variational formulations of the speeds of propagation which are given in Theorems 1.8 and 1.9 are more general than those in Hamel [6] and Heinze et al. [9]. In Theorems 1.8 and 1.9, we consider nonhomogeneous nonlinearities $f = f(x, y, u)$ and the domain Ω is in the most general periodic situation. However, in [6], the domain was an infinite cylinder of $\mathbb{R}^N$ and the advection q was in the form of shear flows. Moreover, in this paper, the nonhomogeneous operator $\nabla \cdot (A\nabla u)$ replaces the Laplace operator Δu taken in [6]. On the other hand, in [9], the domain Ω was an infinite cylinder in $\mathbb{R}^N$ with a bounded cross section. Namely, $\Omega = \mathbb{R} \times \omega \subset \mathbb{R}^N$ where the cross section ω is a bounded domain in $\mathbb{R}^{N-1}$. Moreover, the authors did not consider an advection field in [9]. Finally, concerning the nonlinearities, they were depending only on u (i.e $f = f(u)$ and is satisfying either (1.11) or (1.12)) in both of [6] and [9].

Besides the fact that we consider here a wider family of diffusion and reaction coefficients, our assumptions are less strict than those supposed in [9] and [16]. Roughly speaking, the authors, in [9] and [16], assume a stability condition on the pulsating travelling fronts. We mention that such a stability condition is fulfilled in the homogenous setting; however, it has not been rigorously proved so far that this condition is satisfied in the heterogenous setting. Meanwhile, the assumptions of the present paper only involve the coefficients of the reaction–advection–diffusion equation (1.14), and they can then be checked easily.

Actually, in the "KPP" case, another "simpler" variational formula for the minimal speed $c^*(e) = c^*_{\Omega,A,q,f}(e)$ is known. This known formula involves only the linearized nonlinearity f at $u = 0$. Namely, it follows from [2] that

Theorem 1.11 (Berestycki et al. [2]) *Let e be a fixed unit vector in $\mathbb{R}^d$ and let $\tilde{e} = (e, 0, \dots, 0) \in \mathbb{R}^N$. Assume that f is a "KPP" nonlinearity and that Ω, A and q satisfy* (1.2), (1.3) *and* (1.4)*, respectively. Then, the minimal speed $c^*(e)$ of pulsating fronts solving* (1.14) *and propagating in the direction of e is given by*

$$c^*(e) = c^*_{\Omega,A,q,f}(e) = \min_{\lambda>0} \frac{k(\lambda)}{\lambda}, \tag{1.18}$$

where $k(\lambda) = k_{\Omega,e,A,q,\zeta}(\lambda)$ is the principal eigenvalue of the operator $L_{\Omega,e,A,q,\zeta,\lambda}$ which is defined by

$$\begin{aligned} L_{\Omega,e,A,q,\zeta,\lambda}\psi := \nabla \cdot (A\nabla\psi) + 2\lambda\tilde{e} \cdot A\nabla\psi + q \cdot \nabla\psi \\ + \left[\lambda^2 \tilde{e}A\tilde{e} + \lambda\nabla \cdot (A\tilde{e}) + \lambda q \cdot \tilde{e} + \zeta\right]\psi \end{aligned} \tag{1.19}$$

acting on the set

$$\widetilde{E_\lambda} = \big\{\psi \in C^2(\overline{\Omega}),\ \psi \text{ is } L\text{-periodic with respect to } x \text{ and } \\ \nu \cdot A\nabla\psi = -\lambda(\nu \cdot A\tilde{e})\psi \text{ on } \partial\Omega\big\}.$$

In our last result, we prove that formula (1.17) implies formula (1.18) in the "KPP" case, but under some additional assumptions on the advection and the diffusion coefficients. This result gives an alternate proof of the formula (1.18).

Theorem 1.12 *Let e be a fixed unit vector in $\mathbb{R}^d$ and let $\tilde{e} = (e, 0, \dots, 0) \in \mathbb{R}^N$. Assume that f is a "KPP" nonlinearity and that Ω, A and q satisfy* (1.2), (1.3) *and* (1.4)*, respectively. Assume, furthermore, that $\nu \cdot A\tilde{e} = 0$ on $\partial\Omega$ (in the case where $\partial\Omega \neq \emptyset$). Then, formula* (1.17) *implies formula* (1.18).

2 Main tools: change of variables and maximum principles

In this section, we introduce some tools that will be used in different places of this paper in order to prove the main results.

Throughout this paper, $\tilde{e}$ will denote the vector in $\mathbb{R}^N$ defined by

$$\tilde{e} = (e, 0, \dots, 0) = (e^1, \dots, e^d, 0, \dots, 0),$$

where $e^1, \dots, e^d$ are the components of the vector e.

Our study is concerned with the model (1.14). Having a "combustion", a "ZFK", or a "KPP" nonlinearity, together with the assumptions (1.3) and (1.4), problem (1.14) has at least a classical solution (c, u) such that $c > 0$ and $u_t > 0$ (see Theorems 1.5 and 1.6). The function u is globally $C^{1,\mu}(\mathbb{R} \times \overline{\Omega})$ and $C^{2,\mu}$ with respect to (x, y) variables (for every $\mu \in [0, 1)$). It follows that $\nabla_{x,y}.(A\nabla u) \in C(\mathbb{R} \times \overline{\Omega})$. Having a unit direction $e \in \mathbb{R}^d$, and having a bounded classical solution (c, u) of (1.14) with $c = c(e)$ (combustion case) or $c \geq c^*(e)$ (ZFK or KPP case), we make the same change of variables as Xin [19]. Namely, let $\phi = \phi(s, x, y)$ be the function defined by

$$\phi(s, x, y) = u\left(\frac{s - x \cdot e}{c}, x, y\right) \quad \text{for all } s \in \mathbb{R} \text{ and } (x, y) \in \overline{\Omega}. \tag{2.1}$$

Then, for all $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$,

$$\begin{aligned}&\left[\nabla_{x,y} \cdot (A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \nabla_{x,y} \cdot (A\tilde{e}\phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi)\right](s, x, y)\\ &\quad = \nabla_{x,y} \cdot (A\nabla u)(t, x, y),\end{aligned}$$

where $s = x \cdot e + ct$. Consequently,

$$F[\phi](s, x, y) = \nabla_{x,y} \cdot (A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \nabla_{x,y} \cdot (A\tilde{e}\phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi)$$

is defined at each point $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$ and the map $(s, x, y) \mapsto F[\phi](s, x, y)$ belongs to $C(\mathbb{R} \times \overline{\Omega})$.

In all this paper, $L = L_c$ will denote the operator acting on the set E (given in Theorem 1.8) and which is defined by

$$\begin{aligned}L\varphi &= \nabla_{x,y} \cdot (A\nabla_{x,y}\varphi) + (\tilde{e}A\tilde{e})\varphi_{ss} + \nabla_{x,y} \cdot (A\tilde{e}\varphi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\varphi)\\ &\qquad + q \cdot \nabla_{x,y}\varphi + (q \cdot \tilde{e} - c)\varphi_s \quad \text{in } C(\mathbb{R} \times \overline{\Omega})\\ &= F[\varphi] + q \cdot \nabla_{x,y}\varphi + (q \cdot \tilde{e} - c)\varphi_s \quad \text{in } C(\mathbb{R} \times \overline{\Omega}),\end{aligned} \tag{2.2}$$

for all $\varphi \in E$.

It follows from above that if $\phi = \phi(s, x, y)$ is a function that is given by a pulsating travelling (c, u) solving (1.14) (under the change of variables (2.1)), then $F[\phi] \in C(\mathbb{R} \times \overline{\Omega})$, ϕ is globally bounded in $C^{1,\mu}(\mathbb{R} \times \overline{\Omega})$ (for every $\mu \in [0, 1)$) and it satisfies the following degenerate elliptic equation

$$\begin{aligned}L\phi(s, x, y) + f(x, y, \phi) &= F[\phi](s, x, y) + q \cdot \nabla_{x,y}\phi(s, x, y)\\ &\qquad + (q \cdot \tilde{e} - c)\phi_s(s, x, y) + f(x, y, \phi) = 0\end{aligned} \tag{2.3}$$

in $\mathbb{R} \times \overline{\Omega}$, together with the boundary and periodicity conditions

$$\begin{cases}\phi \text{ is } L\text{-periodic with respect to } x,\\ \nu \cdot A(\nabla_{x,y}\phi + \tilde{e}\phi_s) = 0 \quad \text{on } \mathbb{R} \times \overline{\Omega}.\end{cases} \tag{2.4}$$

Moreover, since $u(t, x, y) \to 0$ as $x \cdot e \to -\infty$ and $u(t, x, y) \to 1$ as $x \cdot e \to +\infty$ locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e, and since ϕ is L-periodic with respect to x, the change of variables $s = x \cdot e + ct$ guarantees that

$$\phi(-\infty, \cdot, \cdot) = 0 \quad \text{and} \quad \phi(+\infty, \cdot, \cdot) = 1 \text{ uniformly in } (x, y) \in \overline{\Omega}. \tag{2.5}$$

Therefore, one can conclude that $\phi \in E$.

Remark 2.1 It is now clear that a max-min formula of the type (1.16) can not hold for the minimal speed $c^*(e) > 0$ in the "ZFK" or the "KPP" case. Indeed, for each speed $c \geq c^*(e)$, there is a solution (c, u) of (1.14) such that $u_t > 0$, which gives birth to a function $\phi = \phi(s, x, y)$ under the change of variables (2.1). Owing to the above discussions the function $\phi \in E$ and it satisfies

$$c = R\phi(s, x, y) \quad \text{for all } (s, x, y) \in \mathbb{R} \times \overline{\Omega}.$$

Therefore

$$\sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y) \geq c.$$

Since one can choose any $c \geq c^*(e)$, one concludes that

$$\sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y) = +\infty$$

in the "ZFK" or the "KPP" case.

Remark 2.2 (*The same formulæ, but over a subset of E*) If the restriction of the nonlinear source f in (1.14) is $C^{1,\delta}(\overline{\Omega} \times [0, 1])$, one can then conclude that (see the proof of Proposition 6.3 in [1]) any solution u of (1.14) satisfies:

$$\forall (t, x, y) \in \mathbb{R} \times \overline{\Omega}, \quad |\partial_{tt} u(t, x, y)| \leq M\, \partial_t u(t, x, y)$$

for some constant M independent of (t, x, y). In other words, the function

$$\phi(s, x, y) = u((s - x \cdot e)/c, x, y)$$

(where $c = c(e)$ in the "combustion" case, and $c = c^*(e)$ in the "ZFK" or the "KPP" case) satisfies

$$\forall (s, x, y) \in \mathbb{R} \times \overline{\Omega}, \quad |\partial_{ss}\phi(s, x, y)| \leq (M/c)\, \partial_s \phi(s, x, y).$$

Let $E^{'}$ be the functional subset of E defined by

$$E^{'} = \left\{\varphi \in E,\ \exists C > 0,\ \forall\, (s, x, y) \in \mathbb{R} \times \overline{\Omega},\ |\partial_{ss}\varphi(s, x, y)| \leq C\, \partial_s \varphi(s, x, y)\right\}.$$

The previous facts together with the discussions at the beginning of this section imply that the functions ϕ and ϕ^* of Theorems 1.8 and 1.9 are elements of $E^{'} \subset E$. These theorems also yield that the max–min and the min–max formulæ can also hold over the subset $E^{'}$ of E.

Namely, in the case of a "combustion" nonlinearity

$$c(e) = \min_{\varphi \in E^{'}} \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y) \tag{2.6}$$

and

$$c(e) = \max_{\varphi \in E^{'}} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\,\varphi(s, x, y). \tag{2.7}$$

Moreover, the min and the max are attained at, and only at, the function $\phi(s, x, y)$ and its shifts $\phi(s + \tau, x, y)$ for any $\tau \in \mathbb{R}$.

On the other hand, only a min–max formula holds in the case of "ZFK" or "KPP" non-linearities. That is

$$c^*(e) = \min_{\varphi \in E'} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\,\varphi(s, x, y). \tag{2.8}$$

Moreover, the min is attained at the function $\phi^*(s, x, y)$ and its shifts $\phi^*(s + \tau, x, y)$ for any $\tau \in \mathbb{R}$.

In the proofs of the variational formulæ which were given in Theorem 1.8 and Theorem 1.9, we will use two versions of the maximum principle in unbounded domains for some problems related to (2.2)–(2.4) and (2.5). Such generalized maximum principles were proved in Berestycki and Hamel [1] in a slightly more general framework:

Lemma 2.3 [1] *Let e be a fixed unit vector in $\mathbb{R}^d$. Let $g(x, y, u)$ be a globally bounded and globally Lipschitz-continuous function defined in $\overline{\Omega} \times \mathbb{R}$ and assume that g is non-increasing with respect to u in $\overline{\Omega} \times (-\infty, \delta]$ for some $\delta > 0$. Let $h \in \mathbb{R}$ and $\Sigma_h^- := (-\infty, h) \times \Omega$. Let $c \neq 0$ and $\phi^1(s, x, y)$, $\phi^2(s, x, y)$ be two bounded and globally $C^{1,\mu}(\overline{\Sigma_h^-})$ functions (for some $\mu > 0$) such that*

$$\begin{cases} L\phi^1 + g(x, y, \phi^1) \geq 0 & \text{in } \mathcal{D}'(\Sigma_h^-),\\ L\phi^2 + g(x, y, \phi^2) \leq 0 & \text{in } \mathcal{D}'(\Sigma_h^-),\\ \nu \cdot A\left[\tilde{e}(\phi_s^1 - \phi_s^2) + \nabla_{x,y}(\phi^1 - \phi^2)\right] \leq 0 & \text{on } (-\infty, h] \times \partial\Omega,\\ \lim_{s_0\to-\infty} \sup_{\{s\leq s_0,\ (x,y)\in\overline{\Omega}\}} [\phi^1(s, x, y) - \phi^2(s, x, y)] \leq 0, \end{cases} \tag{2.9}$$

where

$$\begin{aligned} L\phi := {} & \nabla_{x,y} \cdot (A\nabla_{x,y}\phi) + (\tilde{e}A\tilde{e})\phi_{ss} + \nabla_{x,y} \cdot (A\tilde{e}\phi_s) + \partial_s(\tilde{e}A\nabla_{x,y}\phi)\\ & + q \cdot \nabla_{x,y}\phi + (q \cdot \tilde{e} - c)\phi_s, \end{aligned} \tag{2.10}$$

and $\tilde{e}$ denotes the vector $(e, 0, \ldots, 0) \in \mathbb{R}^N$.

If $\phi^1 \leq \delta$ in $\overline{\Sigma_h^-}$ and $\phi^1(h, x, y) \leq \phi^2(h, x, y)$ for all $(x, y) \in \overline{\Omega}$, then

$$\phi^1 \leq \phi^2 \quad \text{in } \overline{\Sigma_h^-}.$$

Remark 2.4 Note here that ϕ^1, ϕ^2, q, A and g are not assumed to be L-periodic in x and that q is not assumed to satisfy (1.4).

Changing $\phi^1(s, x, y)$, $\phi^2(s, x, y)$ and $g(x, y, s)$ into $1 - \phi^1(-s, x, y)$, $1 - \phi^2(-s, x, y)$ and $-g(x, y, 1 - s)$ respectively in Lemma 2.3 leads to the following

Lemma 2.5 [1] *Let e be a fixed unit vector in $\mathbb{R}^d$. Let $g(x, y, u)$ be a globally bounded and globally Lipschitz-continuous function defined in $\overline{\Omega} \times \mathbb{R}$ and assume that g is non-increasing with respect to u in $\overline{\Omega} \times [1 - \delta, +\infty)$ for some $\delta > 0$. Let $h \in \mathbb{R}$ and $\Sigma_h^+ := (h, +\infty) \times \Omega$. Let $c \neq 0$ and $\phi^1(s, x, y)$, $\phi^2(s, x, y)$ be two bounded and globally $C^{1,\mu}(\overline{\Sigma_h^+})$ functions* *(for some $\mu > 0$) such that*

$$\begin{cases} L\phi^1 + g(x, y, \phi^1) \geq 0 & \textit{in } \mathcal{D}'(\Sigma_h^+), \\ L\phi^2 + g(x, y, \phi^2) \leq 0 & \textit{in } \mathcal{D}'(\Sigma_h^+), \\ \nu \cdot A\left[\tilde{e}(\phi_s^1 - \phi_s^2) + \nabla_{x,y}(\phi^1 - \phi^2)\right] \leq 0 & \textit{on } [h, +\infty) \times \partial\Omega, \\ \lim_{s_0 \to +\infty} \sup_{\{s \geq s_0,\ (x,y) \in \overline{\Omega}\}} [\phi^1(s, x, y) - \phi^2(s, x, y)] \leq 0, \end{cases} \tag{2.11}$$

where L is the same operator as in Lemma 2.3.

If $\phi^2 \geq 1 - \delta$ in $\overline{\Sigma_h^+}$ and $\phi^1(h, x, y) \leq \phi^2(h, x, y)$ for all $(x, y) \in \overline{\Omega}$, then

$$\phi^1 \leq \phi^2 \quad \textit{in } \overline{\Sigma_h^+}.$$

3 Case of a "combustion" nonlinearity

This section is devoted to prove Theorem 1.8, where the nonlinearity f satisfies the assumptions (1.5), (1.6) and (1.7).

3.1 Proof of formula (1.15)

Having a prefixed unit direction $e \in \mathbb{R}^d$, and since the coefficients A and q of problem (1.14) satisfy the assumptions (1.3) and (1.4), it follows, from Theorem 1.5, that there exists a unique pulsating travelling front $(c(e), u)$ (u is unique up to a translation in the time variable) which solves problem (1.14). Moreover, $\partial_t u > 0$ in $\mathbb{R} \times \overline{\Omega}$. We will complete the proof of (1.15) via two steps.

Step 1. After the discussions done in the Sect. 2, the existence of a classical solution $(c(e), u)$, satisfying (1.14), implies the existence of a globally $C^1(\mathbb{R} \times \overline{\Omega})$ function $\phi(s, x, y)$ satisfying $0 \leq \phi \leq 1$ in $\mathbb{R} \times \overline{\Omega}$, with

$$\begin{cases} \phi \text{ is } L\text{-periodic with respect to } x, \\ L\phi(s, x, y) + f(x, y, \phi) = 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \overline{\Omega}), \\ \nu \cdot A(\nabla_{x,y}\phi + \tilde{e}\phi_s) = 0 \text{ in } \mathbb{R} \times \partial\Omega, \\ \phi(-\infty, \cdot, \cdot) = 0, \text{ and } \phi(+\infty, \cdot, \cdot) = 1 \text{ uniformly in } (x, y) \in \overline{\Omega}, \end{cases} \tag{3.1}$$

where L is the operator defined in (2.2) for $c = c(e)$. We also recall that the two functions u and ϕ satisfy the relation

$$u(t, x, y) = \phi(x \cdot e + c(e)t, x, y), \quad (t, x, y) \in \mathbb{R} \times \overline{\Omega}.$$

One has $\partial_s\phi > 0$ in $\mathbb{R} \times \overline{\Omega}$ and this is equivalent to say that the function $u = u(t, x, y)$ is increasing in t, since $c(e) > 0$.

Together with the facts in Sect. 2.1, one gets that the function $\phi \in E$. Furthermore, (3.1) yields that

$$\forall\, s \in \mathbb{R}, \quad \forall (x, y) \in \overline{\Omega}, \quad c(e) = R\,\phi(s, x, y), \tag{3.2}$$

and

$$L\phi(s, x, y) + f(x, y, \phi) = 0, \tag{3.3}$$

where $R\phi$ is the function defined in Theorem 1.8. In other words, the L-periodic (with respect to x) function $R\phi$ is constant over $\mathbb{R}\times\overline{\Omega}$ and it is equal to $c(e)$.

It follows, from (3.2) and from the above explanations, that

$$c(e) \geq \inf_{\varphi\in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y).$$

To complete the proof of formula (1.15), we assume that

$$c(e) > \inf_{\varphi\in E} \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y).$$

Then, there exists a function $\psi = \psi(s,x,y) \in E$ such that

$$c(e) > \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\psi\,(s,x,y).$$

Since the function $\psi \in E$, one then has $\psi_s(s,x,y) > 0$ for all $(s,x,y)\in\mathbb{R}\times\overline{\Omega}$. This yields that

$$L\psi(s,x,y) + f(x,y,\psi) < 0 \quad \text{in } \mathbb{R}\times\overline{\Omega}, \tag{3.4}$$

where L is the operator defined in (2.2) for $c = c(e)$.

Notice that the later holds for each function of the type

$$\psi^\tau(s,x,y) := \psi(s+\tau,x,y)$$

because of the invariance of (3.4) with respect to s and because the advection field q and the diffusion matrix A depend on the variables (x,y) only. That is

$$L\psi^\tau(s,x,y) + f(x,y,\psi^\tau) < 0 \quad \text{in } \mathbb{R}\times\overline{\Omega}. \tag{3.5}$$

Step 2. In order to draw a contradiction, we are going to slide the function ψ with respect to ϕ. From the limiting conditions satisfied by these two functions, there exists a real number $B > 0$ such that

$$\begin{cases} \phi(s,x,y) \leq \theta & \text{for all } s \leq -B,\ (x,y)\in\overline{\Omega},\\ \psi(s,x,y) \geq 1-\rho & \text{for all } s \geq B,\ (x,y)\in\overline{\Omega}, \end{cases}$$

and

$$\phi(B,x,y) \geq 1-\rho \quad \text{for all } (x,y)\in\overline{\Omega}, \tag{3.6}$$

where θ and ρ are the values that appear in the conditions (1.7) satisfied by the "combustion" nonlinearity f. Taking $\tau \geq 2B$, and since ψ is increasing with respect to s, one gets that $\phi(-B,x,y) \leq \psi^\tau(-B,x,y)$ for all $(x,y)\in\overline{\Omega}$ and $\psi^\tau \geq 1-\rho$ in $\overline{\Sigma^+_{-B}}$.

It follows from Lemma 2.3 (take $\delta = \theta,\ h = -B,\ \phi^1 = \phi,$ and $\phi^2 = \psi^\tau$) that $\phi \leq \psi^\tau$ in $\overline{\Sigma^-_{-B}}$. Moreover, Lemma 2.5 (take $\delta = \rho,\ h = -B,\ \phi^1 = \phi,$ and $\phi^2 = \psi^\tau$) implies that $\phi \leq \psi^\tau$ in $\overline{\Sigma^+_{-B}}$. Consequently, $\phi \leq \psi^\tau$ in $\mathbb{R}\times\overline{\Omega}$ for all $\tau \geq 2B$.

Let us now decrease τ and set

$$\tau^* = \inf\left\{\tau\in\mathbb{R},\ \ \phi \leq \psi^\tau \text{ in } \mathbb{R}\times\overline{\Omega}\right\}.$$

First one notes that $\tau^* \leq 2B$. On the other hand, the limiting conditions $\psi(-\infty,\cdot,\cdot) = 0$ and $\phi(+\infty,\cdot,\cdot) = 1$ imply that τ^* is finite. By continuity, $\phi \leq \psi^{\tau^*}$ in $\mathbb{R}\times\overline{\Omega}$. Two cases may occur according to the value of $\sup_{[-B,B]\times\overline{\Omega}}(\phi - \psi^{\tau^*})$.

Case 1: Suppose that

$$\sup_{[-B,B]\times\overline{\Omega}} (\phi - \psi^{\tau^*}) < 0.$$

Since the functions ψ and ϕ are globally $C^1(\mathbb{R}\times\overline{\Omega})$ there exists $\eta > 0$ such that the above inequality holds for all $\tau \in [\tau^* - \eta, \tau^*]$. Choosing any τ in the interval $[\tau^* - \eta, \tau^*]$, and applying Lemma 2.3 to the functions ψ^τ and ϕ, one gets that

$$\phi(s,x,y) \le \psi^\tau(s,x,y) \quad \text{for all } s \le -B,\ (x,y) \in \overline{\Omega},$$

together with the inequality

$$\phi(s,x,y) < \psi^\tau(s,x,y) \quad \text{for all } s \in [-B,B], \quad \text{and for all } (x,y) \in \overline{\Omega}.$$

Owing to (3.6) and to the above inequality, it follows that

$$\psi^\tau(B,x,y) \ge 1-\rho \quad \text{in } \overline{\Omega}.$$

Moreover, since the function ψ is increasing in s, one gets that $\psi^\tau \ge 1-\rho$ in $\overline{\Sigma_B^+}$. Lemma 2.5, applied to ϕ and ψ^τ in $\overline{\Sigma_B^+}$, yields that

$$\phi(s,x,y) \le \psi^\tau(s,x,y) \quad \text{for all } s \ge B,\ (x,y) \in \overline{\Omega}.$$

As a consequence, one obtains $\phi \le \psi^\tau$ in $\mathbb{R}\times\overline{\Omega}$, and that contradicts the minimality of τ^*. Therefore, case 1 is ruled out.

Case 2: Suppose that

$$\sup_{[-B,B]\times\overline{\Omega}} \left(\phi - \psi^{\tau^*}\right) = 0.$$

Then, there exists a sequence of points (s_n, x_n, y_n) in $[-B,B]\times\overline{\Omega}$ such that

$$\phi(s_n,x_n,y_n) - \psi^\tau(s_n,x_n,y_n) \to 0 \quad \text{as } n \to +\infty.$$

Due to the L-periodicity of the functions ϕ and ψ, one can assume that $(x_n, y_n) \in \overline{C}$. Consequently, one can assume, up to extraction of a subsequence, that $(s_n, x_n, y_n) \to (\bar{s}, \bar{x}, \bar{y}) \in [-B,B]\times\overline{C}$ as $n \to +\infty$. By continuity, one gets $\phi(\bar{s},\bar{x},\bar{y}) = \psi^{\tau^*}(\bar{s},\bar{x},\bar{y})$.

We return now to the variables (t,x,y). Let

$$\begin{aligned} z(t,x,y) &= \phi(x\cdot e + c(e)\,t, x, y) - \psi(x\cdot e + c(e)\,t + \tau^*, x, y) \\ &= u(t,x,y) - \psi(x\cdot e + c(e)\,t + \tau^*, x, y) \quad \text{for all } (t,x,y) \in \mathbb{R}\times\overline{\Omega}. \end{aligned}$$

Since the functions ϕ and ψ are in E, it follows that the function z is globally $C^1(\mathbb{R}\times\overline{\Omega})$ and it satisfies

$$\forall\,(t,x,y) \in \mathbb{R}\times\Omega, \quad \nabla_{x,y}\cdot(A\nabla z)(t,x,y) = F[\phi](s,x,y) - F[\psi^{\tau^*}](s,x,y),$$

where $s = x\cdot e + c(e)t$. Thus, $\nabla_{x,y}\cdot(A\nabla z) \in C(\mathbb{R}\times\overline{\Omega})$. Moreover, the function z is non positive and it vanishes at the point $((\bar{s} - \bar{x}\cdot e)/c(e), \bar{x}, \bar{y})$. It satisfies the boundary condition $\nu\cdot(A\nabla z) = 0$ on $\mathbb{R}\times\partial\Omega$. Furthermore, it follows, from (3.2) and (3.4), that

$$\partial_t z - \nabla_{x,y}\cdot(A\nabla z) + q(x,y)\cdot\nabla_{x,y} z \le f(x,y,\phi) - f(x,y,\psi^{\tau^*}).$$

However, the function f is globally Lipschitz-continuous in $\overline{\Omega}\times\mathbb{R}$; hence, there exists a bounded function $b(t,x,y)$ such that

$$\partial_t z - \nabla_{x,y} \cdot (A\nabla z) + q(x,y) \cdot \nabla_{x,y} z + b(t,x,y)\, z \leq 0 \quad \text{in } \mathbb{R} \times \Omega,$$

with $z(t,x,y) \leq 0$ for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$.

Applying the strong parabolic maximum principle and Hopf lemma, one gets that $z(t,x,y) = 0$ for all $t \leq (\bar{s} - \bar{x} \cdot e)/c(e)$ and for all $(x,y) \in \overline{\Omega}$. On the other hand, it follows from the definition of z and from the L-periodicity of the functions ϕ and ψ that $z(t,x,y) = 0$ for all $(t,x,y) \in \mathbb{R} \times \overline{\Omega}$. Consequently,

$$\phi(s,x,y) = \psi^{\tau^*}(s,x,y) = \psi(s+\tau^*,x,y) \quad \text{for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega}.$$

Referring to Eqs. (3.3) and (3.5), one gets a contradiction. Thus, case 2 is ruled out too, and that completes the proof of the formula (1.15).

Remark 3.1 (*The uniqueness, up to a shift, of the minimizer in* (1.15)) If $\psi \in E$ is a minimizer in (1.15). The above arguments imply that case 2 necessarily occurs, and that ψ is equal to a shift of ϕ. In other words, the minimum in (1.15) is realized by and only by the shifts of ϕ.

3.2 Proof of formula (1.16)

In this subsection, we are going to prove the "max–min" formula of the speed of propagation $c(e)$ whenever the nonlinearity f is of the "combustion" type. The tools and techniques which one uses here are similar to those used in the previous subsection. However, we are going to sketch the proof of formula (1.16) for the sake of completeness.

As it was justified in the previous subsection, one easily gets that

$$c(e) \leq \sup_{\varphi \in E} \inf_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y)$$

and

$$\forall\,(s,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad c(e) = R\phi(s,x,y),$$

where

$$\phi(s,x,y) = u\left((s - x\cdot e)/c(e), x, y\right), \quad \text{for all } (s,x,y) \in \mathbb{R} \times \overline{\Omega},$$

and $u = u(t,x,y)$ is the unique (up to a translation in t) pulsating travelling front solving problem (1.14) and propagating in the speed $c(e)$. We recall that the function $\phi \in E$ (see Sect. 2). It follows that the function ϕ satisfies the following

$$\begin{cases} \phi \text{ is } L\text{-periodic with respect to } x, \\ \phi \text{ is increasing in } s \in \mathbb{R}, \\ L\phi(s,x,y) + f(x,y,\phi) = 0 \text{ in } \mathbb{R} \times \overline{\Omega}, \\ \nu \cdot A(\nabla_{x,y}\phi + \tilde{e}\phi_s) = 0 \text{ in } \mathbb{R} \times \partial\Omega, \\ \phi(-\infty,\cdot,\cdot) = 0, \text{ and } \phi(+\infty,\cdot,\cdot) = 1 \text{ uniformly in } (x,y) \in \overline{\Omega}, \end{cases} \tag{3.7}$$

where L is the operator defined in (2.2) for $c = c(e)$.

Notice that the later holds also for each function of the type

$$\phi^\tau(s,x,y) := \phi(s+\tau,x,y)$$

because of the invariance of (3.8) with respect to s and because the advection field q and the diffusion matrix A depend on the variables (x,y) only.

To complete the proof of formula (1.16), we assume that

$$c(e) < \sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\varphi(s, x, y).$$

Hence, there exists $\psi \in E$ such that

$$c(e) < R\psi(s, x, y), \quad \text{for all } (s, x, y) \in \mathbb{R} \times \overline{\Omega}.$$

Since the function $\psi \in E$, one then has $\psi_s(s, x, y) > 0$ for all $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$. This yields that

$$L\psi(s, x, y) + f(x, y, \psi) > 0 \quad \text{in } \mathbb{R} \times \overline{\Omega}. \tag{3.8}$$

To get a contradiction, we are going to slide the function ϕ with respect to ψ. In fact, the limiting conditions satisfied by ψ and ϕ, which are elements of E, yield that there exists a real positive number B such that

$$\begin{cases} \psi(s, x, y) \le \theta & \text{for all } s \le -B,\ (x, y) \in \overline{\Omega}, \\ \phi(s, x, y) \ge 1 - \rho & \text{for all } s \ge B,\ (x, y) \in \overline{\Omega}, \end{cases}$$

and

$$\psi(B, x, y) \ge 1 - \rho \quad \text{for all } (x, y) \in \overline{\Omega}, \tag{3.9}$$

where θ and ρ are the values appearing in the conditions (1.7) satisfied by the nonlinearity f. Having $\tau \ge 2B$, one applies Lemma 2.3 (taking $\delta = \theta,\ h = -B,\ \phi^1 = \psi,$ and $\phi^2 = \phi^\tau$) and Lemma 2.5 (taking $\delta = \rho,\ h = -B,\ \phi^1 = \psi$, and $\phi^2 = \phi^\tau$) to the functions ϕ^τ and ψ, over the domains Σ^-_{-B} and Σ^+_{-B} respectively, to get that $\psi \le \phi^\tau$ in Σ^-_{-B} and $\psi \le \phi^\tau$ in Σ^+_{-B}. Consequently, one can conclude that

$$\forall \tau \ge 2B, \quad \psi \le \phi^\tau \ \text{ in } \mathbb{R} \times \overline{\Omega}.$$

Let us now decrease τ and set

$$\tau^* = \inf\left\{\tau \in \mathbb{R},\ \ \psi \le \phi^\tau \text{ in } \mathbb{R} \times \overline{\Omega}\right\}.$$

It follows, from the limiting conditions $\psi(+\infty, \cdot, \cdot) = 1$ and $\phi(-\infty, \cdot, \cdot) = 0$, that τ^* is finite. By continuity, we have $\psi \le \phi^{\tau^*}$. In this situation, two cases may occur. Namely,

$$\text{case A:} \quad \sup_{[-B,B] \times \overline{\Omega}} (\psi - \phi^{\tau^*}) < 0,$$

or

$$\text{case B:} \quad \sup_{[-B,B] \times \overline{\Omega}} (\psi - \phi^{\tau^*}) = 0.$$

Imitating the ideas and the skills used in case 1 and case 2 during the proof of formula (1.15), one gets that case A (owing to minimality of τ^*) and case B (owing to (3.7) and (3.8)) are ruled out.

Therefore, the assumption that

$$c(e) < \sup_{\varphi \in E} \inf_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\varphi(s, x, y)$$

is false, and that completes the proof of formula (1.16).

Remark 3.2 (*The uniqueness, up to a shift, of the maximizer in* (1.16)) Similar to what we have already mentioned in Remark 3.1, if $\psi \in E$ is a maximizer in (1.16), then the above arguments yield that case B necessarily occurs, and that ψ is equal to a shift of ϕ. One then concludes that the maximum in (1.16) is realized by, and only by, the shifts of ϕ.

4 Case of "ZFK" or "KPP" nonlinearities: proof of formula (1.17)

This section is devoted to the proof of Theorem 1.9. We assume that the nonlinear source f is of "ZFK" type. Remember that this case includes the class of "KPP" nonlinearities. Namely, $f = f(x, y, u)$ is a nonlinearity satisfying (1.5), (1.6) and (1.8). We will divide the proof of formula (1.17) into three steps:

Step 1. Under the assumptions (1.2), (1.3), and (1.4) on the domain Ω, the diffusion matrix A, and the advection field q respectively, and having a nonlinearity f satisfying the above assumptions, Theorem 1.6 yields that for $c = c^*_{\Omega,A,q,f}(e)$, there exists a solution $u^* = u^*(t, x, y)$ of (1.14) such that $u^*_t(t, x, y) > 0$ for all $(t, x, y) \in \mathbb{R} \times \Omega$. In other words, the function ϕ^* defined by

$$\phi^*(s, x, y) = u^*\left(\frac{s - x \cdot e}{c^*(e)}, x, y\right), \quad (s, x, y) \in \mathbb{R} \times \overline{\Omega}$$

is increasing in $s \in \mathbb{R}$. Owing to Sect. 2, ϕ^* satisfies

$$F[\phi^*] + q \cdot \nabla_{x,y}\phi^* + (q \cdot \tilde{e} - c^*(e))\phi^*_s, + f(x, y, \phi^*) = 0 \text{ in } \mathbb{R} \times \overline{\Omega} \tag{4.1}$$

together with boundary and periodicity conditions

$$\begin{cases} \phi^* \text{ is } L\text{-periodic with respect to } x, \\ \nu \cdot A(\nabla_{x,y}\phi^* + \tilde{e}\phi^*_s) = 0 \text{ on } \mathbb{R} \times \overline{\Omega}. \end{cases} \tag{4.2}$$

Moreover, (4.1) implies that

$$\begin{aligned} &\forall\, (s, x, y) \in \mathbb{R} \times \overline{\Omega}, \\ &\qquad c^*(e) = \frac{F[\phi^*](s, x, y) + q \cdot \nabla_{x,y}\phi^*(s, x, y) + f(x, y, \phi^*)}{\partial_s\phi^*(s, x, y)} + q(x, y) \cdot \tilde{e} \\ &\qquad\quad = R\phi^*(s, x, y), \end{aligned} \tag{4.3}$$

and hence

$$c^*(e) \geq \inf_{\varphi \in E} \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} \frac{F[\varphi](s, x, y) + q \cdot \nabla_{x,y}\varphi + f(x, y, \varphi)}{\partial_s\phi(s, x, y)} + q(x, y) \cdot \tilde{e}.$$

In order to prove equality, we argue by contradiction. Assuming that the above inequality is strict, one can find $\delta > 0$ such that

$$c^*(e) - \delta > \inf_{\varphi \in E} \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} \frac{F[\varphi](s, x, y) + q \cdot \nabla_{x,y}\varphi + f(x, y, \varphi)}{\partial_s\varphi(s, x, y)} + q(x, y) \cdot \tilde{e}. \tag{4.4}$$

To draw a contradiction, we are going to approach the "ZFK" nonlinearity f by a sequence of "combustion" nonlinearities $(f_\theta)_\theta$ and the minimal speed of propagation by the sequence of speeds $(c_\theta)_\theta$ corresponding to the functions $(f_\theta)_\theta$. The details will appear in the next step.

Step 2. Let χ be a $C^1(\mathbb{R})$ function such that $0 \leq \chi \leq 1$ in $\mathbb{R}$, $\chi(u) = 0$ for all $u \leq 1$, $0 < \chi(u) < 1$ for all $u \in (1, 2)$ and $\chi(u) = 1$ for all $u \geq 2$. Assume moreover that χ is non-decreasing in $\mathbb{R}$. For all $\theta \in (0, 1/2)$, let χ_θ be the function defined by

$$\forall u \in \mathbb{R}, \quad \chi_\theta(u) = \chi(u/\theta).$$

The function χ_θ is such that $0 \leq \chi_\theta \leq 1$, $0 < \chi_\theta < 1$ in $(-\infty, \theta]$, $0 < \chi_\theta < 1$ in $(\theta, 2\theta)$ and $\chi_\theta = 1$ in $[2\theta, +\infty)$. Furthermore, the functions χ_θ are non-increasing with respect to θ, namely, $\chi_{\theta_1} \geq \chi_{\theta_2}$ if $0 < \theta_1 \leq \theta_2 < 1/2$.

We set

$$f_\theta(x, y, u) = f(x, y, u)\, \chi_\theta(u) \quad \text{for all } (x, y, u) \in \overline{\Omega} \times \mathbb{R}.$$

In other words, we cut off the source term f near $u = 0$.

For each $\theta \in (0, 1/2)$, the function f_θ is a nonlinearity of "combustion" type that satisfies (1.5), (1.6) and (1.7) with the ignition temperature θ. Therefore, Theorem 1.5 yields that the existence of a classical solution (c_θ, u_θ) of (1.14) with the nonlinearity f_θ. Furthermore, the function u_θ is increasing in t and unique up to translation in t and the speed c_θ is unique and positive.

It was proved, through Lemma 6.1 and Lemma 6.2 in Berestycki and Hamel [1], that the speeds c_θ are non-increasing with respect to θ and

$$c_\theta \nearrow c^*(e) \quad \text{as } \theta \searrow 0.$$

Consider a sequence $\theta_n \searrow 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $c_{\theta_n} \geq c^*(e) - \delta$ for all $n \geq n_0$ (or equivalently $\theta_n \leq \theta_{n_0}$).

In what follows, we fix θ such that $\theta < \theta_{n_0}$. One consequently gets $c_\theta \geq c^*(e) - \delta$. On the other hand, it follows, from the construction of f_θ, that $f \geq f_\theta$ in $\overline{\Omega} \times \mathbb{R}$. Together with (4.4), one obtains

$$c_\theta > \inf_{\varphi \in E} \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} \frac{F[\varphi](s, x, y) + q \cdot \nabla_{x,y}\varphi + f_\theta(x, y, \varphi)}{\partial_s \varphi(s, x, y)} + q \cdot \tilde{e}. \tag{4.5}$$

Thus, there exists a function $\psi \in E$ such that

$$c_\theta > \frac{F[\psi](s, x, y) + q \cdot \nabla_{x,y}\psi(s, x, y) + f_\theta(x, y, \psi)}{\partial_s \psi(s, x, y)} + q(x, y) \cdot \tilde{e}. \tag{4.6}$$

However, $\psi_s(s, x, y) > 0$ for all $(s, x, y) \in \mathbb{R} \times \overline{\Omega}$. Thus, the inequality (4.6) can be rewritten as

$$L\psi(s, x, y) + f_\theta(x, y, \psi) < 0 \quad \text{in } \mathbb{R} \times \Omega, \tag{4.7}$$

with $\psi \in E$ and L is the operator defined in (2.2) for $c = c_\theta$.

For each $\tau \in \mathbb{R}$, we define the function ψ^τ by

$$\psi^\tau(s, x, y) = \psi(s + \tau, x, y) \quad \text{for all } (s, x, y) \in \mathbb{R} \times \overline{\Omega}.$$

Since the coefficients of L are independent of s, the later inequality also holds for all functions ψ^τ with $\tau \in \mathbb{R}$. That is,

$$L\psi^\tau(s, x, y) + f_\theta(x, y, \psi^\tau) < 0 \quad \text{in } \mathbb{R} \times \overline{\Omega}. \tag{4.8}$$

Step 3. For the fixed θ (in step 2), the function f_θ is a "combustion" nonlinearity whose ignition temperature is θ. There corresponds a solution (c_θ, u_θ) of (1.14) within the nonlinear source f_θ. We define ϕ_θ by

$$\phi_\theta(s,x,y) = u_\theta\left(\frac{s - x\cdot e}{c_\theta}, x, y\right), \quad \text{for all } (s,x,y)\in\mathbb{R}\times\overline{\Omega}.$$

Referring to Sect. 2, one knows that $\phi_\theta \in E$ and thus it satisfies the following equation

$$L\phi_\theta(s,x,y) + f_\theta(x,y,\phi_\theta) = 0 \quad \text{in } \mathbb{R}\times\overline{\Omega}. \tag{4.9}$$

Now, the situation is exactly the same as that in step 2 of the proof of formula (1.15) because the nonlinearity f_θ is of "combustion" type. The little difference is that f (in step 2 of the proof of formula (1.15)) is replaced here by f_θ, and the function ϕ of equation (3.3) is replaced by the function ϕ_θ of (4.9). Thus, following the arguments of subsection 3.1 and using the same tools of "step 2" as in the proof of formula (1.15), one gets that the (4.4) is impossible and that completes the proof of formula (1.17). □

Remark 4.1 We found that one can use another argument (details are below) different from the sliding method in order to prove the min–max formulæ for the speeds of propagation whenever f is a homogenous (i.e $f = f(u)$) nonlinearity of "combustion" or "ZFK" type and $\Omega = \mathbb{R}^N$. Meanwhile, the sliding method, that we used in the proofs of formulæ (1.15) and (1.17), is a unified argument that works in the general heterogenous periodic framework.

Another proof of formulæ (1.15) *and* (1.17) *in a particular framework.* Here, we assume that $f = f(u)$, and $\Omega = \mathbb{R}^N$. Following the same procedure of "step 1" in the previous proof, one gets the inequality

$$c^*(e) \geq \inf_{\varphi\in E}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y).$$

Now, to prove the other sense of inequality, we assume that

$$c^*(e) > \inf_{\varphi\in E}\ \sup_{(s,x,y)\in\mathbb{R}\times\overline{\Omega}} R\varphi(s,x,y),$$

and we assume that f is of "ZFK" type.[1] Then, as it was explained in "step 2" of the previous proof, one can find $\psi \in E$, $\delta > 0$, $\theta > 0$, and $d > 0$ such that

$$c^*(e) - \delta < d < c_\theta < c^*(e)$$

where

$$\forall (s,x,y)\in\mathbb{R}\times\overline{\Omega}, \quad d > c^*(e) - \delta > R\psi(s,x,y),$$

and $f_\theta(u) = f(u)\,\chi_\theta(u) \leq f(u)$ for all $u \in \mathbb{R}$ is of "combustion" type (c_θ is the speed of propagation, in the direction of $-e$, of pulsating travelling fronts solving (1.14) with the nonlinearity f_θ and the domain $\Omega = \mathbb{R}^N$).

Hence, for all $(t,x,y)\in\mathbb{R}\times\mathbb{R}^N$,

$$d > \frac{F[\psi](s,x,y) + q\cdot\nabla_{x,y}\psi(s,x,y) + f_\theta(\psi)}{\partial_s\psi(s,x,y)} + q(x,y)\cdot\tilde{e}. \tag{4.10}$$

[1] The case where f is of "combustion" type follows in a similar way.

Let $\tilde{u}(t, x, y) = \psi(x \cdot e + dt, x, y)$. As it was explained in Sect. 2, the function $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \nabla \cdot (A(x, y)\nabla \tilde{u}) - q(x, y) \cdot \nabla \tilde{u} - f_\theta(\tilde{u}) > 0, \ t \in \mathbb{R}, \ (x, y) \in \overline{\Omega}, \\ \nu \cdot A \ \nabla \tilde{u}(t, x, y) = 0, \ t \in \mathbb{R}, \ (x, y) \in \partial\Omega, \\ \forall k \in \prod_{i=1}^d L_i\mathbb{Z}, \ \forall (t, x, y) \in \mathbb{R} \times \overline{\Omega}, \ \tilde{u}(t + \frac{k \cdot e}{d}, x, y) = \tilde{u}(t, x + k, y), \\ 0 \le \tilde{u} \le 1. \end{cases} \tag{4.11}$$

Let $0 \le u_0(x, y) \le 1$ be a function in $C(\mathbb{R}^N)$ such that $u_0(x, y) \to 0$ as $x \cdot e \to -\infty$, and $u_0(x, y) \to 1$ as $x \cdot e \to +\infty$, uniformly in y and all directions of $\mathbb{R}^d$ which are orthogonal to e. Let u be a pulsating front propagating in the direction of $-e$ with the speed c_θ and solving the initial data problem

$$\begin{cases} u_t = \nabla \cdot (A(x, y)\nabla u) + q(x, y) \cdot \nabla u + f_\theta(u), \ t > 0, \ (x, y) \in \overline{\Omega}, \\ u(0, x, y) = u_0(x, y), \\ \nu \cdot A \ \nabla u(t, x, y) = 0, \ t \in \mathbb{R}, \ (x, y) \in \partial\Omega. \end{cases} \tag{4.12}$$

Having $f_\theta(u)$ as a "combustion" nonlinearity, it follows from Xin [18, Theorem 3.5] and Weinberger [17], that

$$\forall r > 0, \quad \lim_{t \to +\infty} \sup_{|x| \le r} u(t, x - cte, y) = 0 \text{ uniformly in } y, \text{ for every } c > c_\theta,$$
$$\text{and} \ \lim_{t \to +\infty} \inf_{|x| \le r} u(t, x - cte, y) = 1 \text{ uniformly in } y, \text{ for every } c < c_\theta. \tag{4.13}$$

This means that the speed of propagation c_θ corresponding to (1.14) is equal to the spreading speed in the direction of $-e$ when the nonlinearity is of "combustion" type and the initial data u_0 satisfies the above conditions.

For all $(t, x, y) \in [0, +\infty) \times \overline{\Omega}$, let $w(t, x, y) = \tilde{u}(t, x, y) - u(t, x, y)$. It follows, from (4.11) and (4.12), that

$$\begin{cases} w_t - \nabla \cdot (A(x, y)\nabla w) - q(x, y) \cdot \nabla w + bw > 0, \ t > 0, \ (x, y) \in \overline{\Omega}, \\ \forall (x, y) \in \overline{\Omega}, \ w(0, x, y) \ge 0, \\ \nu \cdot A \ \nabla w(t, x, y) = 0, \ t \in \mathbb{R}, \ (x, y) \in \partial\Omega, \end{cases} \tag{4.14}$$

for some $b = b(t, x, y) \in C(\mathbb{R} \times \overline{\Omega})$. The parabolic maximum principle implies that $w \ge 0$ in $[0, +\infty) \times \overline{\Omega}$. In other words,

$$\forall (t, x, y) \in [0, +\infty) \times \overline{\Omega}, \quad u(t, x, y) \le \tilde{u}(t, x, y).$$

However, for all $c > d$,

$$\lim_{t \to +\infty} \tilde{u}(t, x - cte, y) = \lim_{t \to +\infty} \psi(x \cdot e + (d - c)t, x - cte, y) = 0$$

locally in x and uniformly in y (since $\psi \in E$). Consequently,

$$\forall c > d, \quad \forall r > 0, \quad \lim_{t \to +\infty} \sup_{|x| \le r} u(t, x - cte, y) = 0 \text{ uniformly in } y.$$

Referring to (4.13), one concludes that $d \ge c_\theta$ which is impossible ($d < c_\theta$). Therefore, our assumption that $c^*(e) > \inf_{\varphi \in E} \sup_{(s,x,y) \in \mathbb{R} \times \overline{\Omega}} R\varphi(s, x, y)$ is false and that completes the proof of formula (1.17) in the case where $f = f(u)$ and $\Omega = \mathbb{R}^N$. □

Acknowledgments I am deeply grateful to Professor François Hamel for his valuable directions and advices during the preparation of this paper. I would like also to thank Professor Andrej Zlatoš for his important comments, given while reading a preprint of this work, which allowed me to consider a wider family of "ZFK" nonlinearities in Theorem 1.9 and thus get a more general result.

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