



Asymptotic Regimes of an Integro-Difference Equation with Discontinuous Kernel

Omar Abdul Halim¹ · Mohammad El Smaily²

Received: 8 November 2021 / Revised: 8 November 2021 / Accepted: 7 February 2023

© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract

This paper is concerned with an integral equation that models discrete time dynamics of a population in a patchy landscape. The patches in the domain are reflected through the discontinuity of the kernel of the integral operator at a finite number of points in the whole domain. We prove the existence and uniqueness of a stationary state under certain assumptions on the principal eigenvalue of the linearized integral operator and the growth term as well. We also derive criteria under which the population undergoes extinction (in which case the stationary solution is 0 everywhere).

Keywords Structured integro-difference equation · Principal eigenvalue · Extinction · Discontinuous kernel · Patchy landscape

Mathematics Subject Classification 92D25 · 92D40 · 45G10 · 39A10

1 Introduction and Main Results

In this paper, we study the long term dynamics of a population in a heterogeneous landscape. The density function of this population obeys an integral equation, which we describe in what follows. The population density in the n th generation, at a location x , is denoted by $u_n(x)$. The growth phase is described by some non-negative function F , and the dispersal phase is described by a dispersal kernel $k(x, y)$. The probability that an individual, who started its dispersal process at x , will settle in $[y, y + dy)$ is then given by the product $k(x, y)dy$. The population density in the next generation, at a location x , is then obtained by summing up arrivals at x from all possible locations y . This yields the integral equation

✉ Mohammad El Smaily
smaily@unbc.ca

Omar Abdul Halim
oabduhl1@ualberta.ca

¹ Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada

² Department of Mathematics and Statistics, University of Northern British Columbia, Prince George, BC V2N 4Z9, Canada

$$u_{n+1}(x) = \int_{\Omega} k(x, y) F(u_n(y)) dy, \quad (1)$$

where $u_n(x)$ stands for the density of the population in the n^{th} generation at a position x .

Let us mention some past works that relate to the above model. Musgrave and Lutscher [9] considered (1) in the case where the domain Ω consists of patches that result a difference in the dispersal behaviour. At the interface of two patch types, the authors of [9] incorporate recent results of Ovaskainen and Cornell [10] that, in general, lead to a discontinuous density function for the random walker. In other words, a discontinuity in the dispersal kernel k at the interface of these patches appears in the study done in [9]. In [9], it is shown that the dispersal kernel can be characterized as the Green's function of a second-order differential operator. Later, Watmough and Beykzadeh [1] made generalization of the classic Laplace kernel, which includes different dispersal rates in each patch as well as different degrees of bias at the patch boundaries. We also mention the work of Lewis et al. [8], which considers the same model (1), but with a different set of assumptions on the kernel k . In [8], the kernel k is assumed to be in the form $k(x, y) = k(x - y)$. The more important difference between our work and [8] is that the kernel k is assumed to be continuous in [8]. In our present work, we allow discontinuity of k at a finite number of points of the domain. For more details on the nature of discontinuity of our kernel, see the assumption (H1) below. Roughly speaking, the discontinuity points account for a different dispersal behaviour; hence, a change in the patch where the population is moving. The discontinuity of k adds more technicality to the proofs of the main results than in [8], especially those involving comparison arguments.

Our model (1) is a discrete time analogue of the (continuous) time-space reaction-diffusion model

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2} + f(u). \quad (2)$$

In (2), t stands for the continuous time variable and x stands for the continuous space variable. A higher dimensional version of (2), which also accounts for an underlying advection and heterogeneity in the landscape, is the semilinear parabolic partial differential equation

$$\frac{\partial u}{\partial t}(t, x) = \Delta u + q(x) \cdot \nabla u + f(u), \quad t > 0, \quad x \in \mathbb{R}^N, \quad (3)$$

for some incompressible vector field $q : \mathbb{R}^N \rightarrow \mathbb{R}^N$. When (3) is used to describe the evolution of a population density $u(t, x)$, it is assumed that the dispersal of the species follows the normal distribution with zero mean, which however is not the case for most species. Equations of the type (3) have been studied in much detail. Traveling fronts, or pulsating traveling fronts, form a particular class of solutions to such equations. The existence and qualitative properties of these travelling fronts and their propagation speeds have been studied in a long list of works (see [3–6] and the references therein).

Equation (1) represents an alternative discrete-time model often used in biological literature. An advantage of the IDE (1) over the reaction-diffusion equation (2) lies in the fact that (1) can be used to model the spatial spread of long-distance dispersers, while (2) is inappropriate in this situation.

In this paper, we study the convergence of the integro-difference equation (1), where $\Omega = (-a, a)$. We make the following assumptions:

(H1) The function $k \geq 0$ is a bounded nonnegative function on $\Omega \times \Omega$, such that

$$\text{for all } (x, y) \in \Omega^2, \quad \delta < k(x, y) \leq \Lambda, \quad (4)$$

for some $\delta, \Lambda > 0$. Moreover, we assume that there exists a finite set of points $\{a_i\}_{1 \leq i \leq n} \subset \Omega$, dividing the domain Ω into intervals

$$\Omega_i = (a_i, a_{i+1}) \text{ for } 1 \leq i \leq n - 1, \quad \Omega_0 = (-a, a_1) \text{ and } \Omega_n = (a_n, a)$$

and that k is continuous on each $\Omega_i \times \Omega_j$ for $1 \leq i, j \leq n$.

(H2) The nonlinearity F satisfies: F is continuous, $0 \leq F(x) \leq M$ for all $x \in \mathbb{R}$.

(H3) The function F is strictly increasing on $[0, \infty)$ and it vanishes elsewhere. We assume that F differentiable at 0 and we set

$$r_0 := F'(0) > 1. \tag{5}$$

Furthermore, the function F satisfies

$$\frac{F(u)}{u} < \frac{F(v)}{v}, \text{ for all } u > v > 0. \tag{6}$$

In this work, we will study the convergence of the sequence $u_{n+1} = \mathcal{T}(u_n)$, where \mathcal{T} is defined by

$$\mathcal{T}(u)(x) = \int_{\Omega} k(x, y)F(u(y))dy, \text{ for } u \in L^2(\Omega). \tag{7}$$

Definition 1 (*Stationary solution*) A measurable function $w: \Omega \rightarrow \mathbb{R}$ is called a stationary solution of (1) if it satisfies $w = \mathcal{T}(w)$.

Definition 2 We denote the linearization of the nonlinear operator \mathcal{T} , at $u = 0$, by

$$\mathcal{T}_0(u)(x) := r_0 \int_{\Omega} k(x, y)u(y)dy, \quad x \in \Omega.$$

We denote by

$$X = \{u \in L^2(\Omega) : u \text{ is continuous on } \Omega \text{ except at finitely many points}\}. \tag{8}$$

We will make use of Krein-Rutman theorem in analyzing the linearized operator \mathcal{T}_0 , introduced in Definition 2. We recall the statement of this theorem, below. The proof of this result can be found in [2].

Theorem A (Krein-Rutman Theorem, [2] Theorem 19.2) *Let E be a Banach space and let $K \subset E$ be a closed convex cone such that $K - K$ is dense in E . Let $\mathcal{A} : E \rightarrow E$ be a non-zero compact linear operator which is positive, meaning that $\mathcal{A}(K) \subset K$, with positive spectral radius $r(\mathcal{A})$. Then, there exists a principal eigenvalue $\lambda_0 = r(\mathcal{A}) > 0$ of \mathcal{A} , and an eigenfunction $u \in K \setminus \{0\}$ of \mathcal{A} such that $\mathcal{A}u = \lambda_0 u$.*

We will apply Theorem A to the operator \mathcal{T}_0 , where we let the Banach space E be $L^2(\Omega)$ and take

$$K = L^2(\Omega)^+,$$

where

$$L^2(\Omega)^+ := \{u \in L^2(\Omega), \text{ such that } u \geq 0 \text{ in } \Omega\}.$$

Proposition 1 *The operator \mathcal{T}_0 maps $L^2(\Omega)$ into itself. Moreover, \mathcal{T}_0 is a positive compact operator and there exists an eigenfunction $\phi_0 \in K \setminus \{0\}$ of \mathcal{T}_0 that corresponds to the principal eigenvalue $\lambda_0 := r(\mathcal{T}_0)$.*

Proof We know that K is a cone and the set $K - K$ is dense in $L^2(\Omega)$. The map \mathcal{T}_0 maps $L^2(\Omega)$ into itself because

$$\begin{aligned} \|\mathcal{T}_0(u)\|_2^2 &= r_0^2 \int_{\Omega} \left(\int_{\Omega} k(x, y)u(y)dy \right)^2 dx \\ &\leq r_0^2 \int_{\Omega} \|k(x, \cdot)\|_2^2 \|u\|_2^2 dx \\ &= r_0^2 \|k\|_2^2 \|u\|_2^2 < \infty. \end{aligned}$$

Since $k \in L^2(\Omega^2)$, it then follows (see Lax [7], for eg.) that \mathcal{T}_0 is a compact linear operator. Moreover, if $u \in K$ and $x \in \Omega$, then

$$\mathcal{T}_0(u)(x) = r_0 \int_{\Omega} k(x, y)u(y) dy \geq r_0\delta \int_{-a}^a u(y) dy \geq 0$$

as $u \geq 0$ and $u \not\equiv 0$. Hence, $\mathcal{T}_0(u) \geq 0$ and $\mathcal{T}_0(u) \in K$. Then, \mathcal{T}_0 is positive.

Since \mathcal{T}_0 is a bounded linear operator, the spectral radius of \mathcal{T} is (see [7])

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_0^n\|^{\frac{1}{n}} = r(\mathcal{T}_0)$$

For $v(x) = \frac{1}{\sqrt{|\Omega|}}$, we have $\|v\|_2 = 1$. Using v , we obtain

$$\begin{aligned} \|\mathcal{T}_0^n\| &= \sup_{\|u\|_2 \leq 1} \|\mathcal{T}_0^n(u)\|_2 \geq \|\mathcal{T}_0^n(v)\|_2 \\ &\geq \left(\int_{\Omega} \left(r_0 \int_{\Omega} k(x, y_1)\mathcal{T}_0^{n-1}(v)(y_1)dy_1 \right)^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega} \left(r_0^n \int_{\Omega} k(x, y_1) \int_{\Omega} k(y_1, y_2) \int_{\Omega} \dots \int_{\Omega} k(y_{n-1}, y_n)v(y_n)dy_n \dots dy_1 \right)^2 dx \right)^{\frac{1}{2}} \\ &\geq r_0^n \left(\delta^{2n} \int_{\Omega} \left(\int_{\Omega} \dots \int_{\Omega} \frac{1}{\sqrt{|\Omega|}} dy_n \dots dy_1 \right)^2 dx \right)^{\frac{1}{2}} \\ &= r_0^n \delta^n \frac{|\Omega|^{n+1}}{|\Omega|} = r_0^n \delta^n |\Omega|^n. \end{aligned}$$

Taking the n th root both sides and the limit as n goes to ∞ we get that

$$r(\mathcal{T}_0) \geq r_0\delta|\Omega| > 0. \tag{9}$$

This proves that the spectral radius is positive (i.e. $r(\mathcal{T}_0) > 0$). We can then apply Theorem A to obtain the existence of an eigenfunction $\phi_0 \in K$ of \mathcal{T}_0 that corresponds to the principal eigenvalue $\lambda_0 := r(\mathcal{T}_0)$. □

Our first two results answer the question about the stationary state of (1) in the case where $F(0) = 0$. The main criterion used to decide the nature of the stationary solution is the value of the principal eigenvalue λ_0 .

Theorem 1 *Let $\{u_n\}_n$ be a solution of (1), with the initial condition $u_0 \in X$. Suppose that F and k satisfy the assumptions (H1), (H2) and (H3). Assume furthermore that $F(0) = 0$. If $\lambda_0 \leq 1$, then*

1. 0 is the only stationary solution of (1) in X .
2. The sequence $\{u_n\}_n$ converges to 0 in $L^2(\Omega)$, as $n \rightarrow \infty$.

Theorem 1 shows that the population will eventually undergo extinction, whenever the principal eigenvalue λ_0 is less or equal 1. The next theorem shows that the threshold $\lambda_0 = 1$ is sharp in the sense that the population settles at a positive stationary state whenever $\lambda_0 > 1$.

As a consequence of Theorem 1, we can now consider the case with mortality. In general, mortality of individuals is included in the dispersal [9]. This can be reflected on the kernel by assuming that

$$\int_{\Omega} k(x, y) \, dy \leq 1 \text{ for all } x \in \Omega. \tag{10}$$

Under the assumption (10), the following result holds:

Corollary 1 *Let $\{u_n\}_n$ be a solution of (1), with the initial condition $u_0 \in X$. Suppose that F and k satisfy the assumptions (H1), (H2), (H3) and the additional assumption (10). Assume furthermore that $F(0) = 0$ and $r_0 \leq 1$. Then,*

1. 0 is the only stationary solution of (1) in X .
2. The sequence $\{u_n\}_n$ converges to 0 in $L^2(\Omega)$, as $n \rightarrow \infty$.

Our third result in the case where $F(0) = 0$ is the following:

Theorem 2 *Suppose that F and k satisfy the assumptions (H1), (H2) and (H3). Assume furthermore that $F(0) = 0$. Suppose that $u_0 \in X$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and that $\lambda_0 > 1$. Then,*

1. There exists a unique positive stationary solution w of (1).
2. The sequence $\{u_n\}_n$ converges to w in $L^2(\Omega)$, as $n \rightarrow \infty$.

In the next theorem, we will see that the stationary state will be positive, regardless of λ_0 , whenever $F(0) > 0$.

Theorem 3 *Suppose that F and k satisfy the assumptions (H1), (H2) and (H3). Assume furthermore that $F(0) > 0$. If $u_0 \in X$, $u_0 \geq 0$ and $u_0 \not\equiv 0$, then*

1. There exists a unique positive stationary solution w of (1).
2. The sequence $\{u_n\}_n$ converges to w in $L^2(\Omega)$, as $n \rightarrow \infty$.

2 Proofs of the Main Results

We start with a series of Lemmas that will be used in the proofs of the theorems that we announced in Sect. 1, above.

Lemma 1 *Let w be a stationary solution of (1). Then, $w \geq 0$. Moreover, if $w \not\equiv 0$, then $w(x) > 0$ for all $x \in \Omega$.*

Proof First we mention that, since F is nonnegative and $k > 0$, then the function w is nonnegative. We prove the claim according to the value of $F(0)$. If $F(0) > 0$, then

$$\forall x \in \Omega, \quad w(x) = \int_{\Omega} k(x, y)F(w(y)) \, dy > 2a\delta F(0) > 0.$$

If $F(0) = 0$, then for $x \in \Omega$, using (4) we have

$$w(x) = \int_{\Omega} k(x, y)F(w(y)) \, dy \geq \delta \int_{\Omega} F(w(y)) \, dy > 0,$$

because if $\int_{\Omega} F(w(y)) \, dy = 0$, then $F(w(y)) = 0$ a.e in Ω . The latter implies that $w(y) = 0$ a.e in Ω . However, as $w \neq 0$, there exists $z \in \Omega$ such that $w(z) > 0$. Then,

$$0 < w(z) = \int_{\Omega} k(z, y)F(w(y)) \, dy = 0$$

as $w(y) = 0$ a.e in Ω . This is a contradiction. □

Lemma 2 *Let $u, v \in X$ (defined in (8)), such that $u \geq T(u)$ and $v \leq T(v)$. If $u > 0$, then $u \geq v$.*

Proof First, we show that $\inf_{x \in \Omega} u(x) > 0$. We will discuss this according to the value of $F(0)$. If $F(0) > 0$, then we have

$$\forall x \in \Omega, \quad u(x) \geq T(u)(x) \geq 2a\delta F(0) > 0.$$

If $F(0) = 0$, then

$$\forall x \in \Omega, \quad u(x) \geq T(u)(x) = \int_{\Omega} k(x, y)F(u(y)) \, dy \geq \delta \int_{\Omega} F(u(y)) \, dy > 0,$$

Note that $\int_{\Omega} F(u(y)) \, dy > 0$, because we can find an interval in Ω where $u > 0$, which implies $F(u) > 0$ on that interval. This proves our claim. Also, we have

$$v \leq T(v) = \int_{\Omega} k(\cdot, y)F(v(y)) \, dy \leq 2a\Lambda M.$$

Since $u, v \in X$, we denote by $\{B_i\}_{1 \leq i \leq p}$ and $\{C_j\}_{1 \leq j \leq q}$ the subintervals of Ω such that u and v are continuous on each B_i and C_j respectively. Let $A_{ij} = B_i \cap C_j$ and define

$$\alpha_{ij} := \inf \{ \alpha > 0, \alpha u(x) \geq v(x) \text{ for all } x \in A_{ij} \}.$$

The infimum exists as the set $\{ \alpha > 0, \alpha u|_{A_{ij}} \geq v|_{A_{ij}} \}$ is nonempty, $\inf_{x \in \Omega} u(x) > 0$ and v is bounded above. It follows from the continuity of u and v over A_{ij} that $\alpha_{ij}u \geq v$ in A_{ij} . Moreover, from the definition of \inf , there exists $x_{ij} \in A_{ij}$ such that $\alpha_{ij}u(x_{ij}) = v(x_{ij})$. Let

$$\alpha_0 = \max_{ij}(\alpha_{ij}) = \alpha_{i_0j_0}.$$

Then, $\alpha_0u \geq v$ in Ω . Denote by $x_0 := x_{i_0j_0}$. We claim that $\alpha_0 \leq 1$, which would then prove our lemma. Suppose to the contrary that $\alpha_0 > 1$. Then,

$$\begin{aligned} 0 &= \alpha_0u(x_0) - v(x_0) \\ &\geq \alpha_0T(u)(x_0) - T(v)(x_0) \\ &= \int_{\Omega} k(x_0, y) [\alpha_0F(u(y)) - F(v(y))] \, dy. \end{aligned}$$

From assumption (6), and since F is increasing, we get

$$\alpha_0F(u(y)) = \alpha_0u(y) \frac{F(u(y))}{u(y)} > \alpha_0u(y) \frac{F(\alpha_0u(y))}{\alpha_0u(y)} \geq F(v(y)) \text{ for all } y.$$

Thus,

$$\int_{\Omega} k(x_0, y) [\alpha_0F(u(y)) - F(v(y))] \, dy > 0,$$

which is contradiction. Hence, $u \geq \alpha_0u \geq v$ in Ω . Therefore, $u \geq v$ in Ω . □

Lemma 2 leads to the uniqueness of a non-zero stationary solution of (1).

Corollary 2 (Uniqueness of the stationary state) *There exists at most one non-zero stationary solution of (1) in X .*

Proof In fact, if there exists two non-zero stationary solutions u and v , we get $u \geq T(u)$ and $v \leq T(v)$. Lemma 1 implies that $u > 0$ and Lemma 2 yields that $u \geq v$. Also, we have $v \geq T(v)$ and $u \leq T(u)$. From Lemma 1, we then get that $v > 0$. Lemma 2 implies that $v \geq u$. Therefore, $u \equiv v$. □

The following lemma will be used in proving the convergence in a certain mode of a sequence $\{u_n\}_n$ satisfying (1). The lemma is announced in terms two types of equicontinuity of a sequence of functions. We will recall the definitions of pointwise and uniform equicontinuity in what follows and then announce the lemma involving these notions.

Definition 3 (*Pointwise equicontinuous*) A sequence of functions $\{f_n\}_n$ is said to be pointwise equicontinuous on a set J if

$$\forall \varepsilon > 0, \forall z \in J, \exists \delta(z) > 0, \text{ such that } |f_n(x) - f_n(z)| < \varepsilon, \tag{11}$$

whenever $x \in J$ and $|x - z| < \delta(z)$.

Definition 4 (*Uniformly equicontinuous*) A sequence of functions $\{f_n\}_n$ is said to be uniformly equicontinuous on a set J if

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } |f_n(x) - f_n(y)| < \varepsilon \tag{12}$$

whenever $x, y \in J$ and $|x - y| < \delta$.

Lemma 3 *Let J be a compact set. If $\{f_n\}_n \subset C(J)$ is pointwise equicontinuous, then $\{f_n\}_n$ is uniformly equicontinuous.*

Proof Let $\varepsilon > 0$. For every $z \in J$, there exists $\delta_z > 0$ such that for all $x \in B_J(z, \delta_z)$, we have $|f_n(x) - f_n(z)| < \varepsilon/2$ for all $n \in \mathbb{N}$, where $B(z, \delta_z)$ is the open ball of center z and radius δ_z and $B_J(z, \delta_z) = J \cap B(z, \delta_z)$. We have

$$\bigcup_{z \in J} B_J\left(z, \frac{\delta_z}{2}\right) = J.$$

By compactness of J , there exist $z_1, \dots, z_m \in J$ such that

$$J = \bigcup_{i=1}^m B_J\left(z_i, \frac{\delta_{z_i}}{2}\right).$$

We take $\delta_0 < \min_{1 \leq i \leq m} \left(\frac{\delta_{z_i}}{2}\right)$. Let $x, y \in J$ such that $|x - y| < \delta_0$. Then, $x \in B_J\left(z_j, \frac{\delta_{z_j}}{2}\right)$, for some $1 \leq j \leq m$. Now,

$$|y - z_j| \leq |x - y| + |x - z_j| < \delta_0 + \frac{\delta_{z_j}}{2} \leq \frac{\delta_{z_j}}{2} + \frac{\delta_{z_j}}{2} = \delta_{z_j}.$$

Thus, $y \in B_J(z_j, \delta_{z_j})$. It then follows that

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_n(z_j)| + |f_n(y) - f_n(z_j)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all $n \in \mathbb{N}$. Hence, the set $\{f_n\}_n$ is uniformly equicontinuous. □

Lemma 4 *If $u_0 \in X$, then the sequence $\{u_n\}_n$, obtained from (1) with an initial condition u_0 , satisfies $u_n \in X$ for all $n \in \mathbb{N}$. Moreover, if $\{u_n\}_n$ converges pointwise to $w \in L^2(\Omega)$, then $w \in X$.*

Proof As $u_0 \in X$, there exists m subintervals $\{A_i\}_{1 \leq i \leq m}$ in Ω , such that u_0 is continuous over each A_i . We claim that for every n , u_n is continuous on Ω except possibly at the points $\{a_i\}_{1 \leq i \leq n}$ (introduced in assumption (H1)) and at the discontinuity points of u_0 , which are finitely many. Let B_{ij} be the interval defined by

$$B_{ij} := \Omega_i \cap A_j.$$

For $x \in B_{ij}$, we have

$$u_1(x) = \int_{\Omega} k(x, y)F(u_0(y)) \, dy = \sum_{i,j} \int_{B_{ij}} k(x, y)F(u_0(y)) \, dy.$$

Since $y \mapsto k(x, y)F(u_0(y))$ is continuous on each B_{ij} , it follows that u_1 is continuous at x as a sum of a finite number of continuous functions. Hence, u_1 is continuous on Ω except possibly at the points mentioned in the claim. Suppose now that the claim is true for all u_i such that $i \leq n$. Then,

$$u_{n+1}(x) = \int_{\Omega} k(x, y)F(u_n(y)) \, dy = \sum_{i,j} \int_{B_{ij}} k(x, y)F(u_n(y)) \, dy.$$

As the function $y \mapsto k(x, y)F(u_n(y))$ is continuous on each B_{ij} , it then follows that u_{n+1} is continuous at x . This proves the first assertion in our Lemma.

We move now to the proof of the second assertion. Suppose that $\{u_n\}_n$ converges pointwise to $w \in L^2(\Omega)$. We claim that w is continuous on Ω except possibly at the points $\{a_i\}_{1 \leq i \leq n}$ and at the points of discontinuity of u_0 . Suppose that there exists $z \in B_{i_0j_0}$ such that w is discontinuous at z , for some i_0 and j_0 . Since $B_{i_0j_0}$ is open, there exists $r > 0$ such that

$$D := [z - r, z + r] \subset B_{i_0j_0}.$$

We have $\{u_n\}_n$ are all continuous on D . Let us show that $\{u_n\}_n$ is pointwise equicontinuous. Let $x_0 \in D$ and $\varepsilon > 0$. The function $(x, y) \mapsto k(x, y)$ is uniformly continuous on $D \times B_{ij}$, for all (i, j) . This is because the function k can be extended to a continuous and bounded function on each of the compact sets $\bar{D} \times B_{ij}$. Thus, there exist $h_{ij} > 0$ such that, for all (x, y) and (x_1, y_1) in $D \times B_{ij}$, we have

$$|(x - x_1, y - y_1)| < h_{ij} \implies |k(x, y) - k(x_1, y_1)| < l := \frac{\varepsilon}{2aM},$$

where M is an upper bound of F . Let $h := \min_{i,j} \{h_{ij}\}$. Then, for all i, j and for all $(x, y) \in D \times B_{ij}$, we have

$$|x - x_0| = |(x, y) - (x_0, y)| < h \implies |k(x, y) - k(x_0, y)| < l.$$

Thus, for every n and for every $x \in D$, such that $|x - x_0| < h$, we have

$$\begin{aligned} |u_n(x) - u_n(x_0)| &< M \int_{-a}^a |k(x, y) - k(x_0, y)| \, dy \\ &\leq M \sum_{i,j} \int_{B_{ij}} |k(x, y) - k(x_0, y)| \, dy \\ &< M \sum_{i,j} \frac{\varepsilon}{2aM} |B_{ij}| = \varepsilon, \end{aligned}$$

where $|B_{ij}|$ denotes the Lebesgue measure of B_{ij} . Hence, $\{u_n\}_n$ is pointwise equicontinuous. From Lemma 3, we conclude that $\{u_n\}_n$ is uniformly equicontinuous because D is compact. Moreover, $\{u_n\}_n \subseteq C(D)$ converges to w pointwise and it is equicontinuous. Therefore, $\{u_n\}_n$ converges uniformly to w ; hence, w is continuous over D . This contradicts the assumption that w is discontinuous at $z \in D$. The claim then follows and $w \in X$. □

Lemma 5 *If $u_1 \geq u_0$ (respectively $u_1 \leq u_0$), then $\{u_n\}_n$ is increasing (respectively decreasing).*

Proof Suppose that, for all $i \leq n - 1$, we have $u_{i+1} \geq u_i$. Then,

$$u_{n+1}(x) = \int_{\Omega} k(x, y)F(u_n(y)) \, dy \geq \int_{\Omega} k(x, y)F(u_{n-1}(y)) \, dy = u_n(x),$$

because F is increasing. The other case can be proven similarly. □

Lemma 6 *If $\{u_n\}_n$ converges pointwise to w , then $\{u_n\}_n$ converges to w in $L^2(\Omega)$.*

Proof Since w is the limit of $\{u_n\}_n$, it is a stationary solution: indeed

$$\mathcal{T}(w) = \mathcal{T}\left(\lim_{n \rightarrow \infty} u_n\right) = \lim_{n \rightarrow \infty} \mathcal{T}(u_n) = \lim_{n \rightarrow \infty} u_{n+1} = w.$$

In the above equalities, we can interchange the limit and the integral by Lebesgue dominated convergence theorem. Thus,

$$\|u_n - w\|_2^2 = \|u_n - \mathcal{T}(w)\|_2^2 = \int_{\Omega} \left(\int_{-a}^a k(x, y)(F(u_{n-1}(y)) - F(w(y))) \, dy \right)^2 \, dx.$$

We have the following first inequality:

$$\begin{aligned} & \left(\int_{-a}^a k(x, y)(F(u_{n-1}(y)) - F(w(y))) \, dy \right)^2 \\ & \leq \left(\int_{-a}^a [\Lambda(M + M)] \, dy \right)^2 \leq 16a^2 M^2 \Lambda^2 \in L^1(\Omega). \end{aligned}$$

Moreover, we have

$$k(x, y) [F(u_{n-1}(y)) - F(w(y))] \leq 2\Lambda M \in L^1(\Omega).$$

Hence, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \|u_n - w\|_2^2 = \int_{\Omega} \left(\int_{-a}^a \lim_{n \rightarrow \infty} k(x, y)(F(u_{n-1}(y)) - F(w(y))) \, dy \right)^2 \, dx = 0.$$

The proof of Lemma 6 is now complete. □

We are now in position to prove the three main theorems announced in Sect. 1.

Proof of Theorem 1 We know that 0 is a stationary solution of \mathcal{T} . We claim that there is no positive stationary solution. Suppose that there exists a positive stationary solution w of \mathcal{T} . It then follows, from Lemma 1, that $w > 0$ on Ω . Using assumption (6), we have

$$F(w(y)) = w(y) \frac{F(w(y))}{w(y)} < r_0 w(y), \text{ for all } y \in \Omega, \tag{13}$$

where $r_0 = F'(0)$. Thus,

$$\begin{aligned}
 w(x) &= \int_{\Omega} k(x, y)F(w(y)) \, dy \\
 &< r_0 \int_{\Omega} k(x, y)w(y) \, dy = \mathcal{T}_0(w)(x),
 \end{aligned}
 \tag{14}$$

for all $x \in \Omega$. We will now prove that there exists $\mu \in (0, 1)$, such that

$$w \leq (1 - \mu)\mathcal{T}_0(w) \text{ in } \Omega. \tag{15}$$

Let $\alpha := \sup_{x \in \Omega} \frac{w(x)}{\mathcal{T}_0(w)(x)}$. Then there exists $x_0 \in \bar{\Omega}$, such that

$$\lim_{x \rightarrow x_0} \frac{w(x)}{\mathcal{T}_0(w)(x)} = \alpha.$$

From (14), we get $0 < \alpha \leq 1$. Our goal is to prove that $0 < \alpha < 1$. Suppose to the contrary that $\alpha = 1$. We then have

$$\lim_{x \rightarrow x_0} w(x) = \lim_{x \rightarrow x_0} \mathcal{T}_0(w)(x). \tag{16}$$

But (13) and Lebesgue dominated convergence theorem lead to

$$\begin{aligned}
 \lim_{x \rightarrow x_0} w(x) &= \lim_{x \rightarrow x_0} \int_{\Omega} k(x, y)F(w(y)) \, dy \\
 &= \int_{\Omega} \lim_{x \rightarrow x_0} k(x, y)F(w(y)) \, dy \\
 &< r_0 \int_{\Omega} \lim_{x \rightarrow x_0} k(x, y)w(y) \, dy \\
 &= \lim_{x \rightarrow x_0} r_0 \int_{\Omega} k(x, y)w(y) \, dy \\
 &= \lim_{x \rightarrow x_0} \mathcal{T}_0(w)(x).
 \end{aligned}$$

This contradicts (16) and thus $\alpha < 1$. This proves the inequality (15). We then iterate to get

$$w \leq (1 - \mu)r_0 \int_{\Omega} k(\cdot, y)w(y) \, dy \leq (1 - \mu)^2 \mathcal{T}_0^2(w) \leq \dots \leq (1 - \mu)^n \mathcal{T}_0^n(w),$$

for all $n \in \mathbb{N}$.

Let us show that $\inf_{\Omega}(\phi_0) > 0$, where ϕ_0 is the principal eigenfunction of \mathcal{T}_0 associated with the principal eigenvalue λ_0 . In fact, since $k(x, y) > \delta$ for all (x, y) , then, for all $x \in \Omega$ we have

$$\phi_0(x) = \frac{r_0}{\lambda_0} \int_{\Omega} k(x, y)\phi_0(y) \, dy > \frac{r_0\delta}{\lambda_0} \int_{\Omega} \phi_0(y) \, dy > 0,$$

because $\phi_0 \not\equiv 0$ and $\phi_0 \geq 0$. Hence,

$$\inf_{\Omega} \phi_0 \geq \frac{r_0\delta}{\lambda_0} \int_{\Omega} \phi_0(y) \, dy > 0.$$

We recall that $\phi_0 > 0$ in Ω and it is unique up to multiplication by a scalar. Thus, we can assume that $w \leq \phi_0$ over Ω since $\inf_{\Omega} \phi_0 > 0$. Therefore,

$$\mathcal{T}_0(w) \leq r_0 \int_{\Omega} k(x, y)w(y) \, dy \leq r_0 \int_{\Omega} k(x, y)\phi_0(y) \, dy = \mathcal{T}_0(\phi_0).$$

Suppose that, for all $i \leq n - 1$, we have $\mathcal{T}_0^i(w) \leq \mathcal{T}_0^i(\phi_0)$. Then,

$$\mathcal{T}_0^n(w) \leq r_0 \int_{\Omega} k(x, y)\mathcal{T}_0^{n-1}(w)(y) dy \leq r_0 \int_{\Omega} k(x, y)\mathcal{T}_0^{n-1}(\phi_0)(y) dy \leq \mathcal{T}_0^n(\phi_0).$$

By induction, and using the assumption that $\lambda_0 \leq 1$, we then obtain that

$$w \leq (1 - \mu)^n \mathcal{T}_0^n(w) \leq (1 - \mu)^n \mathcal{T}_0^n(\phi_0) = (1 - \mu)^n \lambda_0^n \phi_0 \leq (1 - \mu)^n \phi_0,$$

for all $n \in \mathbb{N}$. Since $0 < (1 - \mu) < 1$, passing to the limit as $n \rightarrow +\infty$ in the above inequality yields $w \equiv 0$ in Ω , which is contradiction. This completes the proof of the first result in the theorem.

We move now to the proof of the second result in this theorem. Let $u_0 \in X$ and fix $N > 2a\Lambda M$. Then,

$$\mathcal{T}(N) = \int_{-a}^a k(x, y)F(N) dy \leq 2a\Lambda M < N.$$

From Lemma 5, it then follows that $\{\mathcal{T}^n(N)\}_n$ is decreasing. Since $\{\mathcal{T}^n(N)\}$ is bounded below by 0 uniformly, $\{\mathcal{T}^n(N)\}_n$ converges pointwise to w . Lemma 4 yields the existence of $w \in X$. From the Lebesgue dominated convergence theorem, it follows that

$$\mathcal{T}(w) = \mathcal{T}\left(\lim_{n \rightarrow \infty} \mathcal{T}^n(N)\right) = \lim_{n \rightarrow \infty} \mathcal{T}^{n+1}(N) = w.$$

The above means that w is stationary solution of \mathcal{T} . Since $\lambda_0 \leq 1$, it follows, from the first part of Theorem 1, that $w = 0$. Now,

$$0 \leq u_1 = \mathcal{T}(u_0) \leq 2r\Lambda M < N.$$

Suppose that, for all $i \leq n$, we have $u_i \leq \mathcal{T}^{i-1}(N)$. Then,

$$u_{n+1} = \int_{-a}^a k(\cdot, y)F(u_n(y)) dy \leq \int_{-a}^a k(\cdot, y)F(\mathcal{T}^{n-1}(N)(y)) dy = \mathcal{T}^n(N).$$

By induction, it then follows that

$$0 \leq u_n \leq \mathcal{T}^{n-1}(N) \text{ for all } n \in \mathbb{N}.$$

Taking the limits in the last inequality yields that $\{u_n\}_n$ converges to 0 pointwise. From Lemma 6, it follows that $\{u_n\}_n$ converges to 0 in $L^2(\Omega)$. □

Proof of Corollary 1 From Theorem 1, it suffices to prove that $\lambda_0 \leq 1$. First, we claim that $\varphi \in L^\infty(\Omega)$. We have

$$\begin{aligned} 0 < \lambda_0 \varphi(x) &= \int_{\Omega} k(x, y)\varphi(y) dy \leq \|k(x, \cdot)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\ &\leq 2a\Lambda \|\varphi\|_{L^2(\Omega)} < \infty, \text{ as } \varphi \in L^2(\Omega). \end{aligned} \tag{17}$$

Hence, our claim that $\varphi \in L^\infty(\Omega)$ follows. Choosing φ such that $\|\varphi\|_{L^\infty(\Omega)} = 1$, we then obtain

$$\begin{aligned} \lambda_0 &= \|\lambda_0 \varphi\|_{L^\infty(\Omega)} = \|\mathcal{T}_0(\varphi)\|_{L^\infty(\Omega)} \\ &= \sup_{\Omega} \left| r_0 \int_{\Omega} k(x, y)\varphi(y) dy \right| \leq \sup_{\Omega} r_0 \int_{\Omega} k(x, y) \|\varphi\|_{L^\infty(\Omega)} dy \\ &= \sup_{\Omega} r_0 \int_{\Omega} k(x, y) dy \leq \sup_{\Omega} r_0 = r_0 \leq 1. \end{aligned}$$

This completes the proof of Corollary 1. □

We are now in position to prove Theorem 2.

Proof of Theorem 2 Here, we have the assumption that $\lambda_0 > 1$ and $F(0) = 0$. Appealing to Lemma 1 and Corollary 2, it suffices to find one positive stationary solution of \mathcal{T} .

Let $h > 0$ such that $\frac{r_0}{1+h} > 1$ and $\frac{\lambda_0}{1+h} > 1$. Note that h exists because $r_0 > 1$ and $\lambda_0 > 1$. Since

$$\begin{aligned} \lambda_0\phi_0(x) &= \mathcal{T}_0(\phi_0)(x) = r_0 \int_{-a}^a k(x, y)\phi_0(y) \, dy \\ &\leq r_0 \left(\int_{-a}^a (k(x, y))^2 \, dy \right)^{1/2} \left(\int_{-a}^a (\phi_0(y))^2 \, dy \right)^{1/2} \\ &\leq r_0\sqrt{2a}\Lambda\|\phi_0\|_2, \end{aligned}$$

it follows that ϕ_0 is bounded above by some constant κ for all x .

We claim that the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{F(\varepsilon\phi_0(x))}{\varepsilon\phi_0(x)} = F'(0)$$

is uniform in $x \in \Omega$.

From the definition of $r_0 = F'(0)$, we have that for any $\zeta > 0$, there exists $\delta_0 > 0$, such that

$$\text{for } |t| < \delta_0, \quad \left| \frac{F(t) - F(0)}{t} - r_0 \right| < \zeta.$$

Let $\varepsilon^* > 0$ be such that $\varepsilon^*\phi_0(x) < \varepsilon^*\kappa < \delta_0$. In other words, ε^* is small enough so that δ_0 is a uniform upper bound of $\varepsilon^*\phi_0$. Hence, the claim follows. Thus, for $\zeta = \frac{hr_0}{1+h}$, we can find $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\left| \frac{F(\varepsilon\phi_0(x))}{\varepsilon\phi_0(x)} - r_0 \right| < \frac{hr_0}{1+h}, \quad \text{for all } x \in \Omega.$$

Thus,

$$\frac{F(\varepsilon\phi_0(x))}{\varepsilon\phi_0(x)} > \frac{r_0}{1+h},$$

for all $\varepsilon \in (0, \varepsilon_0]$. Then, for all $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} \mathcal{T}(\varepsilon\phi_0) &= \int_{\Omega} k(\cdot, y)\varepsilon\phi_0(y) \frac{F(\varepsilon\phi_0(y))}{\varepsilon\phi_0(y)} \, dy \\ &\geq \frac{r_0}{1+h} \int_{\Omega} k(\cdot, y)\varepsilon\phi_0(y) \, dy = \frac{1}{1+h} \mathcal{T}_0(\varepsilon\phi_0) \\ &= \frac{\lambda_0}{1+h} \varepsilon\phi_0 \\ &> \varepsilon\phi_0. \end{aligned} \tag{18}$$

We claim that $\inf_{\Omega}(\mathcal{T}(\varepsilon\phi_0) - \varepsilon\phi_0) > 0$. Suppose that $\inf_{\Omega}(\mathcal{T}(\varepsilon\phi_0) - \varepsilon\phi_0) = 0$. Then, there exists $x_0 \in \bar{\Omega}$ such that

$$\lim_{x \rightarrow x_0} \mathcal{T}(\varepsilon\phi_0)(x) = \lim_{x \rightarrow x_0} \varepsilon\phi_0(x) = \alpha.$$

From (18), we obtain

$$\alpha = \lim_{x \rightarrow x_0} \mathcal{T}(\varepsilon\phi_0)(x) \geq \lim_{x \rightarrow x_0} \frac{\lambda_0}{1+h} \varepsilon\phi_0(x) \geq \lim_{x \rightarrow x_0} \varepsilon\phi_0(x) = \alpha.$$

This implies that

$$\frac{\lambda_0}{1+h}\alpha = \alpha;$$

hence, $\alpha = 0$. But,

$$\begin{aligned} 0 = \alpha &= \lim_{x \rightarrow x_0} \mathcal{T}(\varepsilon\phi_0)(x) \\ &= \lim_{x \rightarrow x_0} \int_{\Omega} k(x, y)F(\varepsilon\phi_0(y)) \, dy \geq \delta \int_{\Omega} F(\varepsilon\phi_0(y)) \, dy > 0, \end{aligned}$$

and so we have a contradiction. Therefore, $\inf_{\Omega}(\mathcal{T}(\varepsilon\phi_0) - \varepsilon\phi_0) > 0$.

Since the set of simple functions is dense in $L^\infty(\Omega)$, and $\varepsilon\phi_0 \in L^\infty(\Omega)$, there exists a sequence of simple functions $\{f_n\}_n$ that decreases uniformly to $\varepsilon\phi_0$. Thus, for

$$\zeta_0 := \inf_{\Omega}(\mathcal{T}(\varepsilon\phi_0) - \varepsilon\phi_0) > 0,$$

there exists $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geq n_\varepsilon$, we have $|f_n(x) - \varepsilon\phi_0(x)| < \zeta_0$ for all $x \in \Omega$. Then, as \mathcal{T} is increasing (which follows from the fact that F is increasing), we get

$$\mathcal{T}(f_n) \geq \mathcal{T}(\varepsilon\phi_0) \geq \varepsilon\phi_0 + \zeta_0 > f_n$$

on Ω , for all $n \geq n_\varepsilon$. Since f_n is a simple function, $f_n \in X$. Now, for $n \geq n_\varepsilon$ we consider the sequence $\{\mathcal{T}^m(f_n)\}_m$. Since $\mathcal{T}(f_n) \geq \mathcal{T}(\varepsilon\phi_0) \geq f_n$ on Ω , Lemma 5 implies that $\{\mathcal{T}^m(f_n)\}_m$ is increasing for all $\varepsilon \leq \varepsilon_0$ and for all $n \geq n_\varepsilon$. Thus, for $0 < \varepsilon < \varepsilon_0$, we have

$$\mathcal{T}^m(f_n) = \int_{-a}^a k(\cdot, y)F(\mathcal{T}^{m-1}(f_n)(y)) \, dy \leq 2a\Lambda M,$$

where Λ and M are the bounds of k and F respectively. Hence, $\{\mathcal{T}^m(f_n)\}_m$ is uniformly bounded and increasing. Therefore, there exists $w \in X$ (by Lemma 4) such that $w = \lim_{m \rightarrow \infty} \mathcal{T}^m(f_n)$ pointwise. The function w is a positive stationary solution of \mathcal{T} since

$$\mathcal{T}(w) = \mathcal{T}\left(\lim_{m \rightarrow \infty} \mathcal{T}^m(f_n)\right) = \lim_{m \rightarrow \infty} \mathcal{T}(\mathcal{T}^m(f_n)) = \lim_{m \rightarrow \infty} \mathcal{T}^{m+1}(f_n) = w.$$

Note that we were able to interchange the limit and the integral in the above equation because of the Lebesgue dominated convergence theorem.

We have

$$0 < \delta \int_{\Omega} F(u_0(y)) \, dy \leq \mathcal{T}(u_0) \leq u_1 \leq 2a\Lambda M. \tag{19}$$

We fix $N > 2a\Lambda M$. Then, $\mathcal{T}(N) \leq 2a\Lambda M < N$. Appealing to Lemma 5, we get $\{\mathcal{T}^n(N)\}_n$ is decreasing and it is bounded uniformly by 0. From Lemma 4, there exists $w^* \in X$ such that $\{\mathcal{T}^n(N)\}_n$ converges pointwise to w^* . This means that w^* is a stationary solution of \mathcal{T} . Then, for $\varepsilon \leq \varepsilon_0$, we have

$$\mathcal{T}(f_n) < N, \text{ for all } n \geq n_\varepsilon.$$

We know that F is increasing. By induction, it follows that

$$\mathcal{T}^n(f_n) \leq \mathcal{T}^n(N).$$

Taking the limit on both sides, we get that $w \leq w^*$. From the uniqueness of the non-zero stationary solution of \mathcal{T} in X , which was established in Corollary 2, we have $w^* = w \neq 0$. From (19), we can find $\varepsilon \in (0, \varepsilon_0]$ such that

$$\varepsilon\phi_0 \leq \varepsilon\phi_0 + v \leq u_1 \leq N \text{ in } \Omega,$$

for some $\nu > 0$. Then, there exists n_0 such that, for all $n \geq n_0$, we have $f_n < \varepsilon\phi_0 + \nu$ in Ω . Let $n_1 > \max\{n_0, n_\varepsilon\}$. Then, $f_{n_1} \leq u_1 \leq N$. Suppose that, for all $i \leq n - 1$, we have

$$\mathcal{T}^i(f_{n_1}) \leq u_{i+1} \leq \mathcal{T}^i(N).$$

Then, since F is increasing, we obtain

$$\begin{aligned} \mathcal{T}^n(f_{n_1}) &= \int_{\Omega} k(\cdot, y)F(\mathcal{T}^{n-1}(f_{n_1}(y))) \, dy \\ &\leq \int_{\Omega} k(\cdot, y)F(u_n(y)) \, dy \\ &\leq \int_{\Omega} k(\cdot, y)F(\mathcal{T}^{n-1}(N)(y)) \, dy. \end{aligned}$$

Hence,

$$\mathcal{T}^n(f_{n_1}) \leq u_{n+1} \leq \mathcal{T}^n(N), \text{ for all } n \in \mathbb{N}.$$

By passing to the limit as $n \rightarrow +\infty$, we get

$$w \leq \lim_{n \rightarrow \infty} u_{n+1} \leq w.$$

Therefore, $\{u_n\}_n$ converges pointwise to w , which is the unique positive stationary solution of T over Ω . From Lemma 6, we obtain that $\{u_n\}_n$ converges to w in $L^2(\Omega)$. This completes the proof of Theorem 2. \square

Proof of Theorem 3 In this proof, we use the assumption that $F(0) > 0$. We fix $N > 2a\Delta M$ and note that

$$\mathcal{T}(N) = \int_{-a}^a k(x, y)F(N) \, dy \leq 2a\Delta M < N.$$

Lemma 5 then yields that $\{\mathcal{T}^n(N)\}$ is decreasing. Since $\{\mathcal{T}^n(N)\}$ is bounded below by 0 uniformly, Lemma 4 leads to the existence of $w \in X$, such that $\{\mathcal{T}^n(N)\}_n$ converges pointwise to w . That is, w is stationary solution of \mathcal{T} . From Corollary 2, it follows that w is the unique stationary solution of (1). We have

$$\mathcal{T}(0)(x) = \int_{\Omega} k(x, y)F(0) \, dy \geq 2a\delta F(0) > 0 \text{ for all } x \in \Omega.$$

From Lemma 5, it follows that $\{\mathcal{T}^n(0)\}_n$ is increasing. Since $\{\mathcal{T}^n(0)\}_n$ is uniformly bounded above by N , Lemma 4 then yields the existence of $w_0 \in X$, such that $\{\mathcal{T}^n(0)\}_n$ converges pointwise to w_0 . This means that w_0 is stationary solution of \mathcal{T} . Corollary 2 yields that $w_0 = w$, the unique stationary solution of (1). Also, we have

$$N \geq \mathcal{T}(u_0) \geq 0.$$

Suppose that it holds true that, for all $i \leq n$,

$$\mathcal{T}^{n-1}(N) \geq u_n \geq \mathcal{T}^{n-1}(0).$$

We compute

$$\begin{aligned}
T^n(N)(x) &= \int_{\Omega} k(x, y) F(T^{n-1}(N)(y)) dy \\
&\geq \int_{\Omega} k(x, y) F(u_n(y)) dy \\
&= u_{n+1}(x) \\
&\geq \int_{\Omega} k(x, y) F(T^{n-1}(0)(y)) dy = T^n(0)(x).
\end{aligned}$$

Thus,

$$T^{n-1}(N) \geq u_n \geq T^{n-1}(0),$$

for all $n \in \mathbb{N}$. Taking the limits in the above inequality, we obtain that

$$w \geq \lim_{n \rightarrow \infty} u_n \geq w.$$

Hence $\{u_n\}_n$ converges to w pointwise. From Lemma 6, it follows that $\{u_n\}_n$ converges to w in $L^2(\Omega)$. \square

Acknowledgements Mohammad El Smaily acknowledges partial support from the Natural Sciences and Engineering Research Council of Canada through the NSERC Discovery Grant RGPIN-2017-04313

References

1. Beykzadeh, A., Watmough, J.: An explicit formula for a dispersal kernel in a patchy landscape. preprint (2019). <https://doi.org/10.1101/680256>
2. Deimling, K.: Nonlinear Functional Analysis. Dover Publications, Mineola (2010)
3. El Smaily, M.: The homogenized equation of a heterogeneous reaction-diffusion model involving pulsating traveling fronts. *Commun. Math. Sci.* **9**(4), 1113–1128 (2011)
4. El Smaily, M., Kirsch, S.: The speed of propagation for KPP reaction-diffusion equations within large drift. *Adv. Diff. Eqn.* **3–4**, 361–400 (2011)
5. El Smaily, M., Kirsch, S.: Asymptotics of the KPP minimal speed within large drift. *C. R. Math. Acad. Sci. Paris* **348**(15–16), 857–861 (2010)
6. El Smaily, M., Kirsch, S.: Front speed enhancement by incompressible flows in three or higher dimensions. *Arch. Ration. Mech. Anal.* **213**(1), 327–354 (2014)
7. Lax, P.: *Functional Analysis, Pure and Applied Mathematics* (New York). Wiley-Interscience [John Wiley & Sons], New York (2002)
8. Lewis, M.A., Marculis, N.G., Shen, Z.: Integrodifference equations in the presence of climate change: persistence criterion, travelling waves and inside dynamics. *J. Math. Biol.* **77**(6–7), 1649–1687 (2018)
9. Musgrave, J., Lutscher, F.: Integrodifference equations in patchy landscapes. *J. Math. Biol.* **69**(3), 617–658 (2014)
10. Ovaskainen, O., Cornell, S.J.: Biased movement at a boundary and conditional occupancy times for diffusion processes. *J. Appl. Prob.* **40**(3), 557–580 (2003)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.