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THE SPEED OF PROPAGATION FOR KPP REACTION-DIFFUSION EQUATIONS WITHIN LARGE DRIFT

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Abstract. This paper is devoted to the study of the asymptotic behaviors of the minimal speed of propagation of pulsating travelling fronts solving the Fisher-KPP reaction-advection-diffusion equation within either a large drift, a mixture of large drift and small reaction, or a mixture of large drift and large diffusion. We consider a periodic heterogenous framework and we use the formula of Berestycki, Hamel, and Nadirashvili [3] for the minimal speed of propagation to prove the asymptotics in any space dimension N. We express the limits as the maxima of certain variational quantities over the family of "first integrals" of the advection field. Then, we perform a detailed study in the case $N=2$ which leads to a necessary and sufficient condition for the positivity of the asymptotic limit of the minimal speed within a large drift.

1. Introduction and main results

In this paper, we study the asymptotics of the minimal speed of propagation of pulsating travelling fronts in the presence of a large incompressible advection field. We consider a reaction-advection-diffusion equation

$$\begin{cases} u_t = \nabla\cdot(A(z)\nabla u) \ + M\,q(z)\cdot\nabla u + f(z,u),\ t\in\mathbb{R},\ z\in\Omega, \\ \nu\cdot A\nabla u = 0 \ \text{ on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{1.1}$$

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where ν stands for the unit outward normal on $\partial\Omega$ whenever it is nonempty.

The domain Ω is C^3 nonempty connected open subset of $\mathbb{R}^N$ such that for some integer $1 \leq d \leq N$, and for some $L_1, \ldots, L_d$ positive real numbers, we have

$$\begin{cases} \exists\, R \geq 0\,; \forall\,(x,y) \in \Omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-d},\ |y| \leq R, \\ \forall\,(k_1, \ldots, k_d) \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}, \quad \Omega = \Omega + \displaystyle\sum_{k=1}^{d} k_i e_i, \end{cases} \tag{1.2}$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of $\mathbb{R}^N$. In other words, Ω is bounded in the y-direction and periodic in x. As archetypes of the domain Ω, we may have the whole space $\mathbb{R}^N$ which corresponds to $d = N$ and $L_1, \ldots, L_N$ any array of positive real numbers. We may also have the whole space $\mathbb{R}^N$ with a periodic array of holes or an infinite cylinder with an oscillating boundary. In this periodic situation, we call

$$C = \{(x,y) \in \Omega : x_1 \in (0, L_1), \ldots, x_d \in (0, L_d)\} \tag{1.3}$$

the periodicity cell of Ω. We also give the following definition:

Definition 1.1 (L-periodic fields)**.** *A field $w : \Omega \to \mathbb{R}^N$ is said to be L-periodic with respect to x if $w(x_1 + k_1, \ldots, x_d + k_d, y) = w(x_1, \ldots, x_d, y)$ almost everywhere in Ω, and for all $k = (k_1, \ldots, k_d) \in \prod_{i=1}^{d} L_i\mathbb{Z}$.*

The diffusion matrix $A(x,y) = (A_{ij}(x,y))_{1 \leq i,j \leq N}$ is a *symmetric* $C^{2,\delta}(\overline{\Omega})$ (with $\delta > 0$) matrix field satisfying

$$\begin{cases} A \text{ is } L\text{-periodic with respect to } x, \\ \exists\, 0 < \alpha_1 \leq \alpha_2, \forall (x,y) \in \Omega, \forall\, \xi \in \mathbb{R}^N, \\ \alpha_1 |\xi|^2 \leq \displaystyle\sum_{1 \leq i,j \leq N} A_{ij}(x,y)\xi_i\xi_j \leq \alpha_2 |\xi|^2. \end{cases} \tag{1.4}$$

The underlying advection $q(x,y) = (q_1(x,y), \ldots, q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) vector field satisfying

$$\begin{cases} q \text{ is } L\text{-periodic with respect to } x, \\ \nabla \cdot q = 0 \quad \text{in } \overline{\Omega}, \\ q \cdot \nu = 0 \quad \text{on } \partial\Omega \text{ (when } \partial\Omega \neq \emptyset), \\ \forall\, 1 \leq i \leq d, \quad \int_C q_i \, dx\, dy = 0. \end{cases} \tag{1.5}$$

Concerning the nonlinearity $f = f(x, y, u)$, it is a nonnegative function defined in $\overline{\Omega} \times [0,1]$ such that

$$\begin{cases} f \geq 0,\ f \text{ is } L\text{-periodic with respect to } x, \text{ and of class } C^{1,\delta}(\overline{\Omega} \times [0,1]), \\ \forall\,(x,y) \in \overline{\Omega},\ f(x,y,0) = f(x,y,1) = 0, \\ \exists\,\rho \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall\, 1-\rho \leq s \leq s' \leq 1,\ f(x,y,s)\ \geq\ f(x,y,s'), \\ \forall\, s \in (0,1),\ \exists\,(x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0, \\ \forall\,(x,y) \in \overline{\Omega},\ \zeta(x,y) := f'_u(x,y,0) = \displaystyle\lim_{u \to 0^+} \frac{f(x,y,u)}{u} > 0, \end{cases} \tag{1.6}$$

with the additional "KPP" assumption (referring to [11] by Kolmogorov, Petrovsky, and Piskunov)

$$\forall\,(x,y,s) \in \overline{\Omega} \times (0,1),\ 0 < f(x,y,s) \leq f'_u(x,y,0) \times\ s. \tag{1.7}$$

An archetype of f is $(x,y,u) \mapsto u(1-u)h(x,y)$ defined on $\overline{\Omega} \times [0,1]$, where h is a positive $C^{1,\delta}(\,\overline{\Omega}\,)$ L-periodic function.

In all of this paper, $e \in \mathbb{R}^d$ is a fixed unit vector and $\tilde{e} := (e, 0, \ldots, 0) \in \mathbb{R}^N$. A pulsating travelling front propagating in the direction of $-e$ within a speed $c \neq 0$ is a solution $u = u(t,x,y)$ of (1.1) for which there exists a function ϕ such that $u(t,x,y) = \phi(x \cdot e + ct, x, y)$, ϕ is L-periodic in x, and

$$\lim_{s \to -\infty} \phi(s,x,y) = 0 \text{ and } \lim_{s \to +\infty} \phi(s,x,y) = 1,$$

uniformly in $(x,y) \in \overline{\Omega}$.

In the same setting as in this paper, it was proved in [1] and [3] that for all Ω, A, q, and f satisfying (1.2), (1.4), (1.5), (1.6–1.7) respectively, there exists $c^*_{\Omega,A,q,f}(e)$, called the *minimal speed of propagation*, such that pulsating travelling fronts exist for $c \geq c^*_{\Omega,A,q,f}(e)$. This result extended that of [11], which proved that $c^*(e) = 2\sqrt{f'(0)}$ in a "homogeneous" framework where $f = f(u)$, $A = I_N$ (the identity matrix), and there is no advection q. A variational formula for the minimal speed $c^*_{\Omega,A,q,f}(e)$ involving the principal eigenvalue of an elliptic operator was proved in [3] and [14]. Moreover, El Smaily [5] proved a min-max formula for the minimal speed. Many asymptotic behaviors of the minimal speed within large or small diffusion and reaction coefficients and many homogenized speeds were found in [4] and [6]. In [4], we have the asymptotic behavior of the minimal speed within a mixture of large diffusion and large advection. Precisely, in Theorem 4.1 of [4], it was proved that for all $0 \leq \gamma \leq 1/2$ and under the condition $\nabla \cdot A\tilde{e} \equiv 0$

in Ω,

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,MA,\,M^\gamma q,\,f}(e)}{\sqrt{M}} = 2\sqrt{\fint_C \tilde{e}\cdot A\tilde{e}(x,y)dx\,dy}\,\sqrt{\fint_C \zeta(x,y)dx\,dy}, \quad (1.8)$$

where $\zeta(x,y)$ is given in (1.6). In this paper, we are interested in the asymptotic behavior of $c^*_{\Omega,A,M\,q,f}(e)/M$ as $M\to+\infty$ and also the asymptotic behaviors of the minimal speed within a mixture of large advection and small reaction or large diffusion. In [2], it has been proved that $\liminf\limits_{M\to+\infty} \dfrac{c^*_{\Omega,A,M\,q,f}(e)}{M}$ (respectively lim sup) are finite. Upper and lower bounds of these lim inf and lim sup were given in [2] in terms of the "first integrals" of the advection field q. The family of first integrals of q and two corresponding sub-families will be used in this paper and we recall their definitions below. In Heinze [9], the limit was given in the case of shear flows $q=(q_1(y),0,\ldots,0)$, where $e=(1,0,\ldots,0)$. An interesting result about the existence of the limit of $c^*_{\Omega,A,M\,q,f}(e)$ as $M\to+\infty$ (where $\Omega=\mathbb{R}^N$) and several examples of the advection field in 2D and 3D were given in Corollary 1.3 and Section 3 of [12].

Definition 1.2 (First integrals)**.** *The family of first integrals of q is defined by*

$$\mathcal{I} := \begin{array}{l}\{w\in H^1_{loc}(\Omega),\ w\neq 0,\ w \text{ is } L\text{-periodic in } x, \text{ and}\\ q\cdot\nabla w = 0 \text{ almost everywhere in } \Omega\}\,.\end{array}$$

Having a matrix $A=A(x,y)$ of the type (1.4)*, we also define*

$$\mathcal{I}_1^A := \left\{ w\in\mathcal{I},\ \text{such that} \int_C \zeta w^2 \geq \int_C \nabla w\cdot A\nabla w \right\}, \quad (1.9)$$

and

$$\mathcal{I}_2^A := \left\{ w\in\mathcal{I},\ \text{such that} \int_C \zeta w^2 \leq \int_C \nabla w\cdot A\nabla w \right\}.$$

Remark 1.1 (more about $\mathcal{I}$)**.** The set $\mathcal{I}\cup\{0\}$ is a closed subspace of $H^1_{loc}(\Omega)$. Moreover, one can see that if $w\in\mathcal{I}$ is a first integral of q and $\eta:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function, then $\eta\circ w\in\mathcal{I}$.

The following theorem gives the asymptotic behavior of the minimal speed in the presence of a large advection:

Theorem 1.1. *We fix a unit direction $e\in\mathbb{R}^d$ and assume that the diffusion matrix A and the nonlinearity f satisfy* (1.4), (1.6), *and* (1.7). *Let q be an* *advection field which satisfies* (1.5). *Then*

$$\lim_{M\to+\infty}\frac{c^*_{\Omega,A,M\,q,f}(e)}{M}=\max_{w\,\in\,\mathcal{I}_1^A}\frac{\int_C(q\cdot\tilde{e})\,w^2}{\int_C w^2}. \tag{1.10}$$

The proof of this theorem will be done later in Section 2.

Remark 1.2. It is worth mentioning that the presence of a large advection $M^\gamma q$ $(0\le\gamma\le 1/2)$ has no influence on the limit in (1.8) whenever a large diffusion MA applies. However, the limit in (1.10) depends on A and f via the set $\mathcal{I}_1^A$ and explicitly on the advection q.

Theorem 1.2 (large advection with small reaction or large diffusion)**.** *Assume that* Ω, A, q, *and* f *satisfy* (1.2), (1.4), (1.5), *and* (1.6–1.7) *respectively. Let* $e\in\mathbb{R}^d$ *be any unit direction. For any* $\varepsilon>0$, $B>0$ *we call* $c^*_{\Omega,A,M\,q,\varepsilon f}(e)$ (*respectively* $c^*_{\Omega,B\,A,M\,q,f}(e)$) *the minimal speed of propagation (in the direction of* $-e$*) of the reaction-advection-diffusion equation with an advection field* Mq *and a reaction term* εf (*respectively diffusion term* $B\,A$)*. Then*

$$\lim_{\varepsilon\to0^+}\lim_{M\to+\infty}\frac{c^*_{\Omega,A,M\,q,\varepsilon f}(e)}{M\sqrt{\varepsilon}}=\Big(2\,\frac{\sqrt{\int_C\zeta}}{|C|}\Big)\max_{w\,\in\,\mathcal{I}}\frac{\int_C(q\cdot\tilde{e})\,w}{\sqrt{\int_C\nabla w\cdot A\nabla w}}, \tag{1.11}$$

and

$$\lim_{B\to+\infty}\lim_{M\to+\infty}\frac{c^*_{\Omega,B\,A,M\,q,f}(e)\times\sqrt{B}}{M}=\Big(2\,\frac{\sqrt{\int_C\zeta}}{|C|}\Big)\max_{w\,\in\,\mathcal{I}}\frac{\int_C(q\cdot\tilde{e})\,w}{\sqrt{\int_C\nabla w\cdot A\nabla w}}. \tag{1.12}$$

The proof of this theorem will be done in Subsection 2.2 below. We mention that many difficulties arose, while demonstrating this result, due to the consideration of a *heterogeneous framework*. Roughly speaking, the fact that the growth $f'_u(x,y,0)=\zeta(x,y)$ and the diffusion $A=A(x,y)$ depend on space variables creates a difficulty in choosing a maximizer of the right-hand side of (1.11) which should satisfy many properties (see Step 4 of the proof for details). We mention that the above result was proved in the *homogeneous case* ($\zeta=f'(0)$ and $A=Id$) by Zlatoš [15]. In the present paper, we will give the proof of these asymptotics in a general framework.

Furthermore, in Section 3 of this paper, we will give more details about the family of first integrals $\mathcal{I}$ and about integrals of the form $\int_C(q\cdot\tilde{e})\,w^2$ (where $w\in\mathcal{I}$) in the case where $N=2$. This will give necessary and sufficient conditions, expressed in terms of the nature of the drift q, for the limit (1.10) to be null or not. In this context, we have Theorem 1.3, which will be stated after the following definition.

Definition 1.3. *Assume that $N = 2$ and that Ω and q satisfy* (1.2) *and* (1.5). *Let $x \in \Omega$ be such that $q(x) \neq 0$. The trajectory of q at x is the largest* (*in the sense of inclusion*) *connected differentiable curve $T(x)$ in Ω satisfying:*

(*i*) $x \in T(x)$,
(*ii*) $\forall y \in T(x)$, $q(y) \neq 0$,
(*iii*) $\forall y \in T(x)$, *$q(y)$ is tangent to $T(x)$ at the point y.*

In the following lemma, we describe the family of "unbounded periodic trajectories" of a vector field q. The proof of this lemma will be done in Section 3.

Lemma 1.1 (unbounded periodic trajectories)**.** *Let $T(x)$ be an unbounded periodic trajectory of q in Ω; that is, there exists $\mathbf{a} \in L_1\mathbb{Z} \times L_2\mathbb{Z} \setminus \{0\}$* (*respectively $L_1\mathbb{Z} \times \{0\} \setminus \{0\}$*) *when $d = 2$* (*respectively $d = 1$*) *such that $T(x) = T(x) + \mathbf{a}$. In this case, we say that $T(x)$ is $\mathbf{a}$-periodic. Then, if $T(y)$ is another unbounded periodic trajectory of q, $T(y)$ is also $\mathbf{a}$-periodic.*

Moreover, in the case $d = 1$, $\mathbf{a} = L_1 e_1$. That is, all the unbounded periodic trajectories of q in Ω are $L_1 e_1$-periodic.

Theorem 1.3. *Assume that $N = 2$ and that Ω and q satisfy* (1.2) *and* (1.5) *respectively. The two following statements are equivalent:*

(i) There exists $w \in \mathcal{I}$ such that $\int_C qw^2 \neq 0$.
(ii) There exists a periodic unbounded trajectory $T(x)$ of q in Ω.

Moreover, if (ii) is satisfied and $T(x)$ is $\mathbf{a}$-periodic, then for any $w \in \mathcal{I}$ we have $\int_C q\, w^2 \in \mathbb{R}\mathbf{a}$.

Remark 1.3. The periodicity assumption on the trajectory in (ii) is crucial. Indeed, there may exist unbounded trajectories which are not periodic, even though the vector field q is periodic. Consider the following function ϕ:

$$\phi(x,y) := \begin{cases} e^{-\frac{1}{\sin^2(\pi y)}} \sin(2\pi(x + \ln(y - [y]))) \text{ if } y \notin \mathbb{Z}, \\ 0 \text{ otherwise,} \end{cases}$$

where $[y]$ denotes the integer part of y. This function is C^∞ on $\mathbb{R}^2$, and 1-periodic in x and y. Hence the vector field $q = \nabla^\perp \phi$ is also C^∞, 1-periodic in x and y, and satisfies $\int_{[0,1]\times[0,1]} q = 0$ with $\nabla \cdot q \equiv 0$. A quick study of this vector field shows that the part of the graph of $x \mapsto e^{-x}$ lying between $y = 0$ and $y = 1$ is a trajectory of q, and is obviously unbounded and not periodic. However, there exists no periodic unbounded trajectory for this vector field, so the theorem asserts that for all $w \in \mathcal{I}$ we have $\int_C qw^2 = 0$.

As a direct consequence of Theorem 1.1 and Theorem 1.3, we get the following corollary about the asymptotic behavior of the minimal speed within large drift:

Corollary 1.1. *Assume that $N = 2$ and that Ω, A, q, and f satisfy the conditions* (1.2), (1.4), (1.5), *and* (1.6–1.7) *respectively. Then*

(*i*) *If there exists no periodic unbounded trajectory of q in Ω, then*

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,M\,q,f}(e)}{M} = 0,$$

for any unit direction e.

(*ii*) *If there exists a periodic unbounded trajectory $T(x)$ of q in Ω (which will be* $\mathbf{a}$*-periodic for some vector $\mathbf{a} \in \mathbb{R}^2$), then*

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,M\,q,f}(e)}{M} > 0 \iff \tilde{e}\cdot\mathbf{a} \neq 0. \tag{1.13}$$

We mention that in the case where $d = 1$, we have $\tilde{e} = \pm e_1$. Lemma 1.1 *yields that $\tilde{e}\cdot\mathbf{a} = \pm L_1 \neq 0$. Referring to* (1.13), *we can then write, for $d = 1$,*

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,M\,q,f}(e)}{M} > 0 \iff \tag{1.14}$$

(*there exists a periodic unbounded trajectory $T(x)$ of q in Ω*).

It is worth mentioning that, in the above corollary, the conditions for which the limit is null or not are expressed only in terms of the advection field q and, moreover, it is easy to check whether they are satisfied by q or not.

Remark 1.4. In (ii), the simplest example is when q is a shear flow (i.e., $q(x_1, x_2) = (q_1(x_2), 0)$). In that case, the limit (1.10) is positive if and only if $\tilde{e}$ is not perpendicular to the flow lines of q (this condition means that the first component of $\tilde{e}$ is not zero).

1.1. **Outline of the rest of the paper.** After the statement of the main results in Section 1, we are going to prove, in Section 2, the asymptotics of the minimal speed within large drift in any space dimension N. This section will be divided into two subsections. In the first one, we prove (1.10), which deals with the asymptotic behavior of the speed in the presence of large advection **only** and then, in Subsection 2.2, we prove (1.11), which concerns the asymptotic behavior of the speed in a mixture of large drift and small reaction (or a mixture of large drift and a large diffusion). In Section 3, we prove many auxiliary lemmas which lead to the proof of Theorem 1.3 and Corollary 1.1 in the case where $N = 2$.

2. Proofs of the asymptotic behaviors in any dimension N

Theorems 1.1 and 1.2 were announced for domains $\Omega \subseteq \mathbb{R}^N$, where N could be any dimension. We divide the present section into two subsections. The first subsection deals with the case where we have only a large advection and the second one deals with the case where we have large advection mixed with a small reaction or a large diffusion.

2.1. **Case of large advection (proof of Theorem 1.1).** Here we prove Theorem 1.1. For this, we start with a proposition which will play an important role in the proof. For the sake of simplicity, we suppose that the diffusion matrix $A = A(x,y) = Id$, where Id is the identity matrix of $M_N(\mathbb{R})$. In the case of any matrix A satisfying (1.4), the proof of (1.10) is very similar to that of the case $A = Id$. Indeed, we point out the simple differences in Remark 2.2 below.

Throughout this proof, since the only parameter in $c^*_{\Omega,A,M\,q,f}(e)$ is the factor M in front of the advection q, we will write $c^*(M) := c^*_{\Omega,A,M\,q,f}(e)$. After supposing that $A = Id$, the subsets $\mathcal{I}_1^A$ and $\mathcal{I}_2^A$ are respectively given by

$$\mathcal{I}_1 := \Big\{ w \in \mathcal{I} \text{ such that } \int_C \zeta w^2 \geq \int_C |\nabla w|^2 \Big\}, \tag{2.1}$$

and

$$\mathcal{I}_2 := \Big\{ w \in \mathcal{I} \text{ such that } \int_C \zeta w^2 \leq \int_C |\nabla w|^2 \Big\}. \tag{2.2}$$

Definition 2.1. *We define $g : [0, +\infty) \to \mathbb{R}$ by*

$$g(\lambda) := \sup_{w \in \mathcal{I}} \frac{1}{\int_C w^2} \Big[\int_C (\zeta w^2 - |\nabla w|^2) + \lambda \int_C (q \cdot \tilde{e}) w^2 \Big], \tag{2.3}$$

and we define $h : (0, +\infty) \to \mathbb{R}$ by

$$h(\lambda) := \frac{g(\lambda)}{\lambda}. \tag{2.4}$$

Remark 2.1. We can replace the supremum by a maximum in (2.3). Indeed, consider for a fixed $\lambda > 0$ a maximizing sequence $\{w_n\}_n$ with $\|w_n\|_{L^2(C)} = 1$. We have

$$\Big[\int_C \big(\zeta w_n^2 - |\nabla w_n|^2\big) + \lambda \int_C (q \cdot \tilde{e}) w_n^2 \Big] \xrightarrow[n \to +\infty]{} g(\lambda).$$

The sequence $\{w_n\}_n$ is then bounded in $H^1_{loc}(\Omega)$, and we can extract a subsequence converging weakly in $H^1_{loc}(\Omega)$ and strongly in $L^2_{loc}(\Omega)$ to w_0. We then have $w_0 \in \mathcal{I}$, $\|w_0\|_{L^2(C)} = 1$. Since

$$\liminf_{n\to\infty} \int_C |\nabla w_n|^2 \geq \int_C |\nabla w_0|^2,$$

we then get

$$\Big[\int_C \big(\zeta w_0^2 - |\nabla w_0|^2\big) + \lambda \int_C (q\cdot\tilde{e}) w_0^2\Big] \geq g(\lambda),$$

and by definition of the supremum, the previous inequality is an equality, which means by the way that the weak convergence in $H^1_{loc}(\Omega)$ is in fact a strong convergence.

Proposition 2.1. *The functions g and h satisfy the following properties:*

(i) The function g is convex on $[0,+\infty)$, and moreover, g and h are continuous on their domains and take values in $(0,+\infty)$.

(ii) $h(\lambda) \xrightarrow[\lambda\to+\infty]{} \sup_{w\in\mathcal{I}} \dfrac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}.$

(iii) Either h is convex and decreasing on $(0,+\infty)$ or h attains a global minimum at some point $\lambda_0 > 0$.

(iv) If h is convex decreasing on $(0,+\infty)$, then we have

$$h(\lambda) \xrightarrow[\lambda\to+\infty]{} \max_{w\in\mathcal{I}_1} \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2} = \sup_{w\in\mathcal{I}} \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}.$$

(v) If h attains its minimum at $\lambda_0 > 0$, then we have

$$h(\lambda_0) = \max_{w\in\mathcal{I}_1} \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}. \tag{2.5}$$

Proof of (i). g is the supremum of affine functions, so it is convex and hence continuous. Since $\int_C \zeta > 0$, and since the constant functions belong to $\mathcal{I}$, we have

$$\forall\lambda \geq 0,\ g(\lambda) \geq \frac{\int_C \zeta}{|C|} > 0.$$

Hence, $g(\lambda) > 0$ for any $\lambda \geq 0$. Besides, h is well defined and continuous, and $h(\lambda) > 0$ for any $\lambda > 0$.

Proof of (ii). For each $k \in \mathbb{N}$, we define

$$\mathcal{I}^k := \Big\{w \in \mathcal{I} \text{ such that } \int_C (\zeta w^2 - |\nabla w|^2) \geq -k \int_C w^2\Big\}$$

and

$$h_k(\lambda) := \sup_{w\in\mathcal{I}^k} \frac{1}{\int_C w^2}\Big[\frac{1}{\lambda}\int_C \big(\zeta w^2 - |\nabla w|^2\big) + \int_C (q\cdot\tilde{e})w^2\Big].$$

Obviously, $h_k(\lambda) \leq h(\lambda)$ for any $\lambda > 0$ because the supremum is taken over a smaller set. Moreover, a simple computation gives

$$h_k(\lambda) \xrightarrow[\lambda \to +\infty]{} \sup_{w \in \mathcal{I}^k} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}.$$

Hence, for every $k \in \mathbb{N}$, we have

$$\liminf_{\lambda \to +\infty} h(\lambda) \geq \sup_{w \in \mathcal{I}^k} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2},$$

and since $\mathcal{I} = \bigcup_{k \in \mathbb{N}} \mathcal{I}^k$, we get

$$\liminf_{\lambda \to +\infty} h(\lambda) \geq \sup_{w \in \mathcal{I}} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}.$$

On the other hand,

$$h(\lambda) \leq \frac{\|\zeta\|_\infty}{\lambda} + \sup_{w \in \mathcal{I}} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2},$$

which gives

$$\limsup_{\lambda \to +\infty} h(\lambda) \leq \sup_{w \in \mathcal{I}} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2},$$

and completes the proof of (ii).

Proof of (iii). We know from (ii) that $h(\lambda)$ converges when $\lambda \to +\infty$. Moreover, since $g(0) > 0$ we then have $h(\lambda) \to +\infty$ as $\lambda \to 0$. We distinguish now two different cases:

Case 1: Suppose that for any $\lambda > 0$, we have $h(\lambda) > \lim_{\lambda \to +\infty} h(\lambda)$. Thus, for a fixed $\lambda > 0$, the definition of the limit yields the existence of $\lambda_1 > \lambda$ such that $h(\lambda) > h(\lambda_1)$. Let then w be such that $\|w\|_{L^2(C)}=1$ and

$$h(\lambda) = \frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda} + \int_C (q \cdot \tilde{e}) w^2$$

(the existence of w follows from Remark 2.1). From the definition of h, we can conclude that

$$\frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda} + \int_C (q \cdot \tilde{e}) w^2 = h(\lambda) > h(\lambda_1) \geq \frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda_1} + \int_C (q \cdot \tilde{e}) w^2,$$

which gives

$$\Big(\frac{1}{\lambda} - \frac{1}{\lambda_1}\Big) \int_C (\zeta w^2 - |\nabla w|^2) \geq 0.$$

Having $\lambda < \lambda_1$, we get

$$\int_C (\zeta w^2 - |\nabla w|^2) \geq 0.$$

Thus, for every $\lambda > 0$, the maximum in the definition of $h(\lambda)$ is attained in $\mathcal{I}_1$. Therefore, h can be rewritten in this case as follows:

$$h(\lambda) = \max_{w \in \mathcal{I}_1} \frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda \int_C w^2} + \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}.$$

In this formulation of h, we maximize over $\mathcal{I}_1$. The map

$$\lambda \mapsto \frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda \int_C w^2}$$

is convex when $w \in \mathcal{I}_1$. Hence, h is the supremum of convex functions and is then convex. Moreover, h converges when $\lambda \to +\infty$, and $h(\lambda) > \lim_{+\infty} h$, which, with the convexity of h, implies that h is decreasing on $(0, +\infty)$.

Case 2: There exists $\lambda > 0$ such that $h(\lambda) \leq \lim_{\lambda \to +\infty} h(\lambda)$. By continuity, there exists $\lambda_0 > 0$ such that $h(\lambda_0) = \min_{\lambda > 0} h(\lambda)$.

Proof of (iv). In the case where h is convex and decreasing over $(0, +\infty)$, we know that

$$h(\lambda) = \max_{w \in \mathcal{I}_1} \frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda \int_C w^2} + \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}.$$

Thus,

$$\max_{w \in \mathcal{I}_1} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2} \leq h(\lambda) \leq \frac{\|\zeta\|_\infty}{\lambda} + \max_{w \in \mathcal{I}_1} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}.$$

We conclude that

$$h(\lambda) \xrightarrow[\lambda \to +\infty]{} \max_{w \in \mathcal{I}_1} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}.$$

Proof of (v). We use several claims to prove this last part of Proposition 2.1. The proofs of these claims are postponed until the end.

Claim 1: There exist $w_1 \in \mathcal{I}_1$ and $w_2 \in \mathcal{I}_2$ such that

$$h(\lambda_0) = \frac{\int_C (\zeta w_1^2 - |\nabla w_1|^2)}{\lambda_0 \int_C w_1^2} + \frac{\int_C (q \cdot \tilde{e}) w_1^2}{\int_C w_1^2} = \frac{\int_C (\zeta w_2^2 - |\nabla w_2|^2)}{\lambda_0 \int_C w_2^2} + \frac{\int_C (q \cdot \tilde{e}) w_2^2}{\int_C w_2^2}.$$

Claim 2: If $w_1 \in \mathcal{I}$ and $w_2 \in \mathcal{I}$ are not proportional and

$$h(\lambda_0) = \frac{\int_C (\zeta w_1^2 - |\nabla w_1|^2)}{\lambda_0 \int_C w_1^2} + \frac{\int_C (q \cdot \tilde{e}) w_1^2}{\int_C w_1^2} = \frac{\int_C (\zeta w_2^2 - |\nabla w_2|^2)}{\lambda_0 \int_C w_2^2} + \frac{\int_C (q \cdot \tilde{e}) w_2^2}{\int_C w_2^2},$$

then for any $0 \leq \theta \leq 1$ and $w_\theta := \theta w_1 + (1-\theta) w_2$, we have

$$h(\lambda_0) = \frac{\int_C (\zeta w_\theta^2 - |\nabla w_\theta|^2)}{\lambda_0 \int_C w_\theta^2} + \frac{\int_C (q \cdot \tilde{e}) w_\theta^2}{\int_C w_\theta^2}.$$

Claim 1 gives us $w_1 \in \mathcal{I}_1$ and $w_2 \in \mathcal{I}_2$ realizing the maximum in the definition of $h(\lambda_0)$. If w_1 and w_2 are proportional, then w_1 (respectively w_2) $\in \mathcal{I}_1 \cap \mathcal{I}_2$ and we define $w_0 := w_1$. If not, using claim 2, we know that any convex combination also realizes the maximum in the definition of h. By continuity, there exists $\theta_0 \in [0,1]$ such that $w_0 := w_{\theta_0} \in \mathcal{I}_1 \cap \mathcal{I}_2$ and

$$h(\lambda_0) = \frac{\int_C (\zeta w_0^2 - |\nabla w_0|^2)}{\lambda_0 \int_C w_0^2} + \frac{\int_C (q \cdot \tilde{e}) w_0^2}{\int_C w_0^2},$$

and since $w_0 \in \mathcal{I}_1 \cap \mathcal{I}_2$, we then have $\int_C (\zeta w_0^2 - |\nabla w_0|^2) = 0$ and consequently

$$h(\lambda_0) = \frac{\int_C (q \cdot \tilde{e}) w_0^2}{\int_C w_0^2}.$$

Since $w_0 \in \mathcal{I}_1$, we have

$$h(\lambda_0) \leq \max_{w \in \mathcal{I}_1} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}.$$

On the other hand, by the definition of h we have

$$\begin{aligned} h(\lambda_0) &= \sup_{w \in \mathcal{I}} \frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda_0 \int_C w^2} + \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2} \\ &\geq \max_{w \in \mathcal{I}_1} \frac{\int_C (\zeta w^2 - |\nabla w|^2)}{\lambda_0 \int_C w^2} + \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2} \geq \max_{w \in \mathcal{I}_1} \frac{\int_C (q \cdot \tilde{e}) w^2}{\int_C w^2}. \end{aligned}$$

This ends the proof of (v).

We are left to prove claims 1 and 2.

Proof of Claim 1: Let $\{\lambda_1^p\}_p$ be a sequence such that $\lambda_1^p < \lambda_0$ and $\lambda_1^p \to \lambda_0$ as $p \to +\infty$.

For each $p \in \mathbb{N}$, let $w_1^p \in \mathcal{I}$ be such that $\int_C (w_1^p)^2 = 1$ and

$$h(\lambda_1^p) = \frac{1}{\lambda_1^p} \int_C \left(\zeta (w_1^p)^2 - |\nabla w_1^p|^2 \right) + \int_C (q \cdot \tilde{e}) (w_1^p)^2.$$

From the definition of h and owing to the fact that $h(\lambda_1^p) \geq h(\lambda_0)$, we have

$$\begin{aligned} h(\lambda_1^p) &= \frac{1}{\lambda_1^p} \int_C \left(\zeta (w_1^p)^2 - |\nabla w_1^p|^2 \right) + \int_C (q \cdot \tilde{e}) (w_1^p)^2 \\ &\geq h(\lambda_0) \geq \frac{1}{\lambda_0} \int_C \left(\zeta (w_1^p)^2 - |\nabla w_1^p|^2 \right) + \int_C (q \cdot \tilde{e}) (w_1^p)^2. \end{aligned}$$

However, $\lambda_1^p \leq \lambda_0$. Thus,

$$\int_C (\zeta (w_1^p)^2 - |\nabla w_1^p|^2) \geq 0, \tag{2.6}$$

which means $w_1^p \in \mathcal{I}_1$. Moreover, (2.6) yields that $\{w_1^p\}_p$ is a bounded sequence in $H^1_{loc}(\Omega)$. Therefore, we can extract a subsequence converging weakly in $H^1_{loc}(\Omega)$ and strongly in $L^2_{loc}(\Omega)$ to $w_1 \in \mathcal{I}$. Since the convergence is strong in $L^2_{loc}(\Omega)$, we get $\int_C (w_1)^2 = 1$. Thanks to the continuity of h with respect to λ, we get

$$\frac{1}{\lambda_1^p} \int_C \left(\zeta (w_1^p)^2 - |\nabla w_1^p|^2\right) + \int_C (q \cdot \tilde{e})(w_1^p)^2 \xrightarrow[p\to\infty]{} h(\lambda_0). \tag{2.7}$$

Moreover, we have

$$h(\lambda_0) \geq \frac{1}{\lambda_0} \int_C \left(\zeta (w_1)^2 - |\nabla w_1|^2\right) + \int_C (q \cdot \tilde{e})(w_1)^2 \tag{2.8}$$

and

$$\frac{1}{\lambda_1^p} \int_C \left(\zeta (w_1^p)^2\right) + \int_C (q \cdot \tilde{e})(w_1^p)^2 \xrightarrow[p\to\infty]{} \frac{1}{\lambda_0} \int_C \left(\zeta (w_1)^2\right) + \int_C (q \cdot \tilde{e})(w_1)^2. \tag{2.9}$$

The combination of (2.7), (2.8), and (2.9) gives

$$\limsup_{p\to\infty} \int_C |\nabla w_1^p|^2 \leq \int_C |\nabla w_1|^2.$$

On the other hand, the weak convergence $w_1^p \rightharpoonup w_1$ in $H^1_{loc}(\Omega)$ implies that

$$\liminf_{p\to\infty} \int_C |\nabla w_1^p|^2 \geq \int_C |\nabla w_1|^2.$$

Hence, $\{w_1^p\}_p$ converges strongly in $H^1_{loc}(\Omega)$ to w_1. We then conclude that $w_1 \in \mathcal{I}_1$ (because $\mathcal{I}_1$ is a closed subset of $H^1_{loc}(\Omega)$) and that

$$h(\lambda_0) = \frac{\int_C (\zeta w_1^2 - |\nabla w_1|^2)}{\lambda_0 \int_C w_1^2} + \frac{\int_C (q \cdot \tilde{e}) w_1^2}{\int_C w_1^2}.$$

We can use a similar argument (we take $\lambda_2^p > \lambda_0$ such that $\lambda_2^p \to \lambda_0$ as $p \to +\infty$ and, for each p, we take w_2^p as a maximizer of $h(\lambda_2^p)$) to get $w_2 \in \mathcal{I}_2$ satisfying

$$h(\lambda_0) = \frac{\int_C (\zeta w_2^2 - |\nabla w_2|^2)}{\lambda_0 \int_C w_2^2} + \frac{\int_C (q \cdot \tilde{e}) w_2^2}{\int_C w_2^2}.$$

Proof of Claim 2: Without loss of generality, we suppose that $\int_C w_1^2 = \int_C w_2^2 = 1$. We consider the following functional defined by

$$\forall\, w \in \mathcal{I},\ E_\lambda(w) := \frac{\int_C(\zeta w^2 - |\nabla w|^2)}{\lambda \int_C w^2} + \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}.$$

We have

$$h(\lambda_0) = \max_{w\in\mathcal{I}} E_{\lambda_0}(w) = E_{\lambda_0}(w_1)$$

and thereby, $\forall w \in \mathcal{I}$ we have

$$E'_{\lambda_0}(w_1)w = 0 = \frac{1}{\lambda_0}\int_C(\zeta w_1 w - \nabla w_1\cdot\nabla w) + \int_C (q\cdot\tilde{e})w_1\cdot w - h(\lambda_0)\int_C w_1 w. \tag{2.10}$$

Now, we compute $E_{\lambda_0}(w_\theta)$ explicitly. We have

$$\int_C w_\theta^2 = \theta^2 + (1-\theta)^2 + 2\theta(1-\theta)\int_C w_1 w_2,$$
$$\int_C \zeta w_\theta^2 = \theta^2\int_C \zeta w_1^2 + (1-\theta)^2\int_C \zeta w_2^2 + 2\theta(1-\theta)\int_C \zeta w_1 w_2,$$
$$\int_C |\nabla w_\theta|^2 = \theta^2\int_C |\nabla w_1^2| + (1-\theta)^2\int_C |\nabla w_2|^2 + 2\theta(1-\theta)\int_C \nabla w_1\cdot\nabla w_2,$$
$$\int_C (q\cdot\tilde{e})w_\theta^2 = \theta^2\int_C (q\cdot\tilde{e})w_1^2 + (1-\theta)^2\int_C (q\cdot\tilde{e})w_2^2 + 2\theta(1-\theta)\int_C (q\cdot\tilde{e})w_1 w_2,$$

and using (2.10) with $w = w_2$ we get

$$E_{\lambda_0}(w_\theta) = \frac{\theta^2 h(\lambda_0) + (1-\theta)^2 h(\lambda_0) + 2\theta(1-\theta)h(\lambda_0)\int_C w_1 w_2}{\theta^2 + (1-\theta)^2 + 2\theta(1-\theta)\int_C w_1 w_2}.$$

The denominator is positive because, by assumption, w_1 and w_2 are not proportional, so we can not have equality in the Cauchy-Schwarz inequality. After simplification, we obtain $E_{\lambda_0}(w_\theta) = h(\lambda_0)$. This completes the proof of Proposition 2.1. $\square$

Proof of Theorem 1.1. From the results of [3], it follows that for each $M > 0$, the minimal speed $c^*(M)$ is given by

$$c^*(M) = \min_{\lambda>0}\frac{k(\lambda, M)}{\lambda}, \tag{2.11}$$

where $k(\lambda, M)$ is the principal eigenvalue of the elliptic operator L_λ defined by

$$L_\lambda\psi := \Delta\psi + 2\lambda\tilde{e}\cdot\nabla\psi + M\, q\cdot\nabla\psi + [\lambda^2 + \lambda M\, q\cdot\tilde{e} + \zeta]\psi \text{ in } \Omega, \tag{2.12}$$

acting on the set $E_\lambda = \{\psi = \psi(x,y) \in C^2(\overline{\Omega}) : \psi$ is L-periodic in x and $\nu \cdot \nabla\psi = -\lambda(\nu\cdot\tilde{e})\psi$ on $\partial\Omega\}$. The principal eigenfunction $\psi^{\lambda,M}$ associated to $k(\lambda,M)$ is positive in $\overline{\Omega}$ and it is unique up to multiplication by a nonzero real number. The existence of $k(\lambda,M)$ and $\psi^{\lambda,M}$ for any $(\lambda,M)\in\mathbb{R}\times\mathbb{R}$, and the properties of $k(\lambda,M)$ as a function of M have been studied in [1] and [3]. In particular, the function $\lambda\mapsto k(\lambda,M)$ is convex and $k(\lambda,M)>0$ for all $(\lambda,M)\in(0,+\infty)\times(0,+\infty)$.

We want to study the asymptotic behavior of $M\mapsto c^*(M)/M$ when $M\to+\infty$. For this, we call $\lambda'=\lambda\times M$, $\mu(\lambda',M)=k(\lambda,M)$, and $\psi^{\lambda',M}=\psi^{\lambda,M}$ for each $(\lambda,M)\in(0,+\infty)\times(0,+\infty)$. Referring to formula (2.11), we then get

$$\forall\, M>0,\ \frac{c^*(M)}{M}=\min_{\lambda'>0}\frac{\mu(\lambda',M)}{\lambda'}. \tag{2.13}$$

From the properties of $k(\lambda,M)$, we have that $\lambda'\mapsto\mu(\lambda',M)$ is convex over $(0,+\infty)$ and $\mu(\lambda',M)>0$ for all $(\lambda',M)\in(0,+\infty)\times(0,+\infty)$. Moreover, it follows from above that the functions $\psi^{\lambda',M}$ and $\mu(\lambda',M)$ are respectively the principal eigenfunction and the principal eigenvalue of the problem

$$\left\{\begin{aligned}\mu(\lambda',M)\psi^{\lambda',M}= {}& \Delta\psi^{\lambda',M}+2\frac{\lambda'}{M}\tilde{e}\cdot\nabla\psi^{\lambda',M}+M\,q\cdot\nabla\psi^{\lambda',M}\\ &+\Big[\Big(\frac{\lambda'}{M}\Big)^2+\lambda'\,q\cdot\tilde{e}+\zeta\Big]\psi^{\lambda',M}\ \text{in }\Omega,\\ \nu\cdot\nabla\psi^{\lambda',M}= {}& -\frac{\lambda'}{M}(\nu\cdot\tilde{e})\psi^{\lambda',M}\ \text{on }\partial\Omega\ (\text{whenever }\partial\Omega\neq\emptyset).\end{aligned}\right. \tag{2.14}$$

We take any first integral $w\in\mathcal{I}$ of q, we multiply (2.14) by $\frac{w^2}{\psi^{\lambda',M}}$, and integrate by parts over the periodicity cell C. Using (1.5), the boundary condition on $\psi^{\lambda',M}$, and the fact that $q\cdot\nabla w=0$ almost everywhere in Ω, we obtain that

$$\begin{aligned}\mu(\lambda',M)\int_C w^2 = {}& \int_C\Big(\frac{\nabla\psi^{\lambda',M}}{\psi^{\lambda',M}}\,w\Big)^2-\frac{\lambda'}{M}\int_{\partial C}\nu\cdot\tilde{e}w^2\\ &-2\int_C\Big(\frac{\nabla\psi^{\lambda',M}}{\psi^{\lambda',M}}w\Big)\cdot\Big(\nabla w-\frac{\lambda'}{M}\tilde{e}w\Big)\\ &+\Big(\frac{\lambda'}{M}\Big)^2\int_C w^2+\lambda'\int_C q\cdot\tilde{e}\,w^2+\int_C\zeta w^2.\end{aligned}$$

Notice that the boundary term $\frac{\lambda'}{M}\int_{\partial C}\nu\cdot\tilde{e}w^2$ is equal to $\frac{\lambda'}{M}\int_{\partial C}2w\nabla w\cdot\tilde{e}$. After dividing the previous equation by λ', we get

$$\frac{\mu(\lambda', M)}{\lambda'} \int_C w^2 = \underbrace{\frac{1}{\lambda'} \int_C \Big| \frac{\nabla \psi^{\lambda',M}}{\psi^{\lambda',M}}\, w - \nabla w + \frac{\lambda'}{M} \tilde{e}\, w \Big|^2}_{\geq 0} \tag{2.15}$$
$$+ \int_C (q \cdot \tilde{e})\, w^2 + \frac{1}{\lambda'} \int_C \big[\zeta w^2 - |\nabla w|^2\big],$$

for all $\lambda' > 0$ and $M > 0$ (in the case of a general diffusion matrix, see Remark 2.2). Since (2.15) is true for any $w \in \mathcal{I}$,

$$\forall\, \lambda', M > 0,\ \frac{\mu(\lambda', M)}{\lambda'} \geq h(\lambda') \geq \inf_{\lambda' > 0} h(\lambda').$$

Having (2.13), one then concludes that for any $\lambda', M \in (0, +\infty)$ and for any $w \in \mathcal{I}$ with $\|w\|_{L^2(C)} = 1$,

$$\inf_{\lambda' > 0} h(\lambda') \leq \frac{c^*(M)}{M} \leq \frac{1}{\lambda'} \int_C \Big| \frac{\nabla \psi^{\lambda',M}}{\psi^{\lambda',M}}\, w - \nabla w + \frac{\lambda'}{M} \tilde{e}\, w \Big|^2 + h(\lambda'). \tag{2.16}$$

To complete the proof we need the following:

Lemma 2.1. *Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $M_n \to +\infty$ as $n \to +\infty$. Then, for a fixed $\lambda' > 0$, the sequence $\{\psi^{\lambda', M_n}\}_{n\in\mathbb{N}}$ of principal eigenfunctions of the problem* (2.14) *corresponding to $M = M_n$ converges strongly, in $H^1_{loc}(\Omega)$, to a function $\psi^{\lambda', +\infty} \in H^1_{loc}(\Omega)$ as $n \to +\infty$. Moreover, $\psi^{\lambda', +\infty}$ is a first integral of q and*

$$\lim_{n \to +\infty} \int_C \Big| \frac{\nabla \psi^{\lambda', M_n}}{\psi^{\lambda', M_n}} \psi^{\lambda', +\infty} - \nabla \psi^{\lambda', +\infty} \Big|^2 = 0. \tag{2.17}$$

The proof of this lemma will be postponed until the end of the proof of Theorem 1.1.

Now, we consider any sequence $\{M_n\}_{n\in\mathbb{N}}$ in $(0, +\infty)$ such that $M_n \to +\infty$ as $n \to +\infty$. Before going further, we mention that parts (iv) and (v) of Proposition 2.1 yield that

$$\max_{w \in \mathcal{I}_1} \frac{\int_C q \cdot \tilde{e}\, w^2}{\int_C w^2} = \inf_{\lambda' > 0} h(\lambda'). \tag{2.18}$$

Together with (2.16), we consequently have

$$\max_{w \in \mathcal{I}_1} \frac{\int_C q \cdot \tilde{e}\, w^2}{\int_C w^2} \leq \liminf_{n \to +\infty} \frac{c^*(M_n)}{M_n}. \tag{2.19}$$

Part (iii) of Proposition 2.1 and (2.16) lead us to two different cases according to the nature of the function h. The first case is when h is convex and decreasing on $(0,+\infty)$. We apply (2.16) for $w=\psi^{\lambda',+\infty}$ (where $\lambda'>0$ is arbitrarily chosen) together with (2.17) and we get

$$\limsup_{n\to+\infty}\frac{c^*(M_n)}{M_n}\le\frac{1}{\lambda'}\lim_{n\to+\infty}\int_C\left|\frac{\nabla\psi^{\lambda',M_n}}{\psi^{\lambda',M_n}}\,\psi^{\lambda',+\infty}-\nabla\psi^{\lambda',+\infty}\right|^2+h(\lambda')=h(\lambda')$$

by (2.17). Since this is true for any $\lambda'>0$, then

$$\limsup_{n\to+\infty}\frac{c^*(M_n)}{M_n}\le\lim_{\lambda\to+\infty}h(\lambda)=\max_{w\in\mathcal{I}_1}\frac{\int_C q\cdot\tilde e\,w^2}{\int_C w^2}\tag{2.20}$$

by part (iv) of the proposition. From (2.19) and (2.20), we get the result in the first case.

The second case is when the function h attains its minimum at $\lambda_0>0$. We apply (2.16) and (2.17) for $\lambda'=\lambda_0$, $w=\psi^{\lambda_0,+\infty}$, and $M=M_n$. Hence,

$$\limsup_{n\to+\infty}\frac{c^*(M_n)}{M_n}=h(\lambda_0)=\min_{\lambda'>0}h(\lambda').$$

Part (v) of Proposition 2.1 together with (2.19) (which is true in both cases) yield that

$$\lim_{n\to+\infty}\frac{c^*(M_n)}{M_n}=\max_{w\in\mathcal{I}_1}\frac{\int_C q\cdot\tilde e\,w^2}{\int_C w^2}$$

in the second case.

Thus, in both cases, the limit of $c^*(M_n)/M_n$ is the same. Moreover, this limit is obtained for an arbitrarily chosen sequence $\{M_n\}_n$ converging to $+\infty$ as $n\to+\infty$. This implies that $\lim_{M\to+\infty}c^*(M)/M$ exists and is equal to $\max_{w\in\mathcal{I}_1}\dfrac{\int_C q\cdot\tilde e\,w^2}{\int_C w^2}$, which eventually proves Theorem 1.1. □

Now, we turn to proving Lemma 2.1, which was stated and used in the proof of Theorem 1.1.

Proof of Lemma 2.1. Let us fix $\lambda'>0$ and take any sequence of positive real numbers $\{M_n\}_{n\in\mathbb{N}}$ converging to $+\infty$ as $n\to+\infty$. For any $n\in\mathbb{N}$, the principal eigenfunction ψ^{λ',M_n} is unique up to multiplication by a nonzero constant. Hence, we can assume that

$$\forall\,n\in\mathbb{N},\ \int_C\left(\psi^{\lambda',M_n}\right)^2=1.\tag{2.21}$$

We multiply (2.14) (where $M=M_n$) by ψ^{λ',M_n} and we integrate by parts over the periodicity cell C. Owing to the periodicity of q and ζ together with the condition (1.5), we then get

$$-\int_C |\nabla\psi^{\lambda',M_n}|^2 + \left(\frac{\lambda'}{M_n}\right)^2 + \int_C \left[\lambda' q\cdot\tilde{e}+\zeta\right]\left(\psi^{\lambda',M_n}\right)^2 = \mu(\lambda',M_n) \quad (2.22)$$

for all $n\in\mathbb{N}$. As direct consequences of (2.22), we have (from a certain rank n_0)

$$\forall n\geq n_0,\ 0<\mu(\lambda',M_n)\leq \lambda'^2+\lambda'\|q\cdot\tilde{e}\|_\infty+\|\zeta\|_\infty, \qquad (2.23)$$

and

$$\forall n\geq n_0,\ 0<\frac{\mu(\lambda',M_n)}{\lambda'}\leq \lambda'+\|q\cdot\tilde{e}\|_\infty+\frac{\|\zeta\|_\infty}{\lambda'}.$$

In other words, the sequences $\{\mu(\lambda',M_n)\}_{n\in\mathbb{N}}$ and $\{\frac{\mu(\lambda',M_n)}{\lambda'}\}_{n\in\mathbb{N}}$ are bounded whenever $\lambda'>0$ is fixed. Thus there exists $\mu(\lambda',+\infty)\geq 0$ such that, up to extraction of a subsequence,

$$\mu(\lambda',M_n)\to\mu(\lambda',+\infty) \text{ as } n\to+\infty. \qquad (2.24)$$

Equations (2.21) and (2.22) imply that the sequence $\{\psi^{\lambda',M_n}\}_{n\in\mathbb{N}}$ is bounded in $H^1(C)$. It follows that there exists $\psi^{\lambda',+\infty}\in H^1_{loc}(\Omega)$ such that, up to extraction of a subsequence, $\psi^{\lambda',M_n}\to\psi^{\lambda',+\infty}$ in $H^1_{loc}(\Omega)$ weakly, in $L^2_{loc}(\Omega)$ strongly, and almost everywhere in Ω as $n\to+\infty$. Thus, the function $\psi^{\lambda',+\infty}$ is L-periodic with respect to x. The strong convergence in $L^2(C)$ leads to $\int_C(\psi^{\lambda',+\infty})^2=1$ and hence $\psi^{\lambda',+\infty}\not\equiv 0$ in Ω. We replace M by M_n in (2.14), divide the equation by M_n, and pass to the limit as $n\to+\infty$ in the sense of distributions. From the weak convergence of $\{\psi^{\lambda',M_n}\}_{n\in\mathbb{N}}$ and the boundedness of $\{\mu(\lambda',M_n)\}_{n\in\mathbb{N}}$, one then has $q\cdot\nabla\psi^{\lambda',+\infty}=0$ in $\mathcal{D}'(\Omega)$. Consequently, $q\cdot\nabla\psi^{\lambda',+\infty}=0$ almost everywhere in Ω. That is, $\psi^{\lambda',+\infty}$ is a nonzero first integral of q.

Now, we multiply (2.14) (where $M=M_n$) by $\psi^{\lambda',+\infty}$, we integrate by parts over C, and we pass to the limit as $n\to+\infty$. We notice that

$$\begin{aligned}\int_C \Delta\psi^{\lambda',M_n}\psi^{\lambda',+\infty} &= -\int_C \nabla\psi^{\lambda',M_n}\cdot\nabla\psi^{\lambda',+\infty} - \frac{\lambda'}{M_n}\int_{\partial C}\nu\cdot\tilde{e}\,\psi^{\lambda',M_n}\psi^{\lambda',+\infty}\\ &\to -\int_C \left|\nabla\psi^{\lambda',+\infty}\right|^2\end{aligned}$$

as $n\to+\infty$ (from the strong convergence in $L^2_{loc}(\Omega)$, the boundary term converges to 0; we use the weak convergence in $H^1_{loc}(\Omega)$ for the limit of the first term), and

$$\int_C \left(q\cdot\nabla\psi^{\lambda',M_n}\right)\psi^{\lambda',+\infty}$$

$$= -\int_C (\nabla \cdot q)\ \psi^{\lambda',+\infty}\psi^{\lambda',M_n} - \int_C \left(q \cdot \nabla\psi^{\lambda',+\infty}\right)\psi^{\lambda',M_n}$$
$$= 0 \text{ (from (1.5) and since } \psi^{\lambda',+\infty} \in \mathcal{I}).$$

Hence,

$$\mu(\lambda', +\infty) = -\int_C \left|\nabla\psi^{\lambda',+\infty}\right|^2 + \int_C \left[\lambda' q \cdot \tilde{e} + \zeta\right]\left(\psi^{\lambda',+\infty}\right)^2. \tag{2.25}$$

On the other hand, we take $w = \psi^{\lambda',+\infty}$ and $M = M_n$ in (2.15), we multiply the equation by the fixed $\lambda' > 0$, and we pass to the limit as $n \to +\infty$ to obtain

$$\begin{aligned} D := \lim_{n\to+\infty}\int_C \Big|\frac{\nabla\psi^{\lambda',M_n}}{\psi^{\lambda',M_n}}\,\psi^{\lambda',+\infty} - \nabla\psi^{\lambda',+\infty} + \frac{\lambda'}{M_n}\tilde{e}\,\psi^{\lambda',+\infty}\Big|^2 \\ = \mu(\lambda',+\infty) - \lambda'\int_C q\cdot\tilde{e}\,(\psi^{\lambda',+\infty})^2 - \int_C \left[\zeta{\psi^{\lambda',+\infty}}^2 - |\nabla\psi^{\lambda',+\infty}|^2\right]. \end{aligned} \tag{2.26}$$

However,

$$\frac{\lambda'}{M_n}\tilde{e}\,\psi^{\lambda',+\infty} \to 0 \ \text{ in } L^2(C) \text{ strongly as } n \to +\infty.$$

Also, $\{|\nabla\psi^{\lambda',M_n}/\psi^{\lambda',M_n}|\}_{n\in\mathbb{N}}$ is bounded in $L^2(C)$. (We simply divide (2.14) by ψ^{λ',M_n} and integrate by parts over C. This leads to

$$\int_C \Big|\frac{\nabla\psi^{\lambda',M_n}}{\psi^{\lambda',M_n}}\Big|^2 + \Big(\frac{\lambda'}{M_n}\Big)^2|C| + \int_C \zeta = \mu(\lambda', M_n)|C|;$$

hence, $\left\{\int_C \Big|\frac{\nabla\psi^{\lambda',M_n}}{\psi^{\lambda',M_n}}\Big|^2\right\}_n$ is bounded due to (2.23).) Consequently,

$$D = \lim_{n\to+\infty}\int_C \Big|\frac{\nabla\psi^{\lambda',M_n}}{\psi^{\lambda',M_n}}\,\psi^{\lambda',+\infty} - \nabla\psi^{\lambda',+\infty}\Big|^2. \tag{2.27}$$

Referring to (2.25), we finally obtain

$$\lim_{n\to+\infty}\int_C \Big|\frac{\nabla\psi^{\lambda',M_n}}{\psi^{\lambda',M_n}}\,\psi^{\lambda',+\infty} - \nabla\psi^{\lambda',+\infty}\Big| = 0.$$

Since $\psi^{\lambda',M_n} \to \psi^{\lambda',+\infty}$ in $L^2_{loc}(\Omega)$ strongly, (2.22) yields that $\int_C |\nabla\psi^{\lambda',M_n}|^2$ converges to

$$\lambda'\int_C q\cdot\tilde{e}\left(\psi^{\lambda',+\infty}\right)^2 + \int_C \zeta\left(\psi^{\lambda',+\infty}\right)^2 - \mu(\lambda',+\infty).$$

Equation (2.25) again yields that

$$\int_C |\nabla\psi^{\lambda',M_n}|^2 \to \int_C |\nabla\psi^{\lambda',+\infty}|^2 \text{ as } n \to +\infty. \tag{2.28}$$

Eventually, $\{\psi^{\lambda',M_n}\}_n$ converges to $\psi^{\lambda',+\infty}$ in $H^1_{loc}(\Omega)$ strongly, and this completes the proof of Lemma 2.1. □

Remark 2.2 (about the proof of (1.10) in case of a diffusion $A = A(x,y)$)**.** In this remark, we detail some differences which arise in the proof of (1.10) when we consider, instead of the identity matrix, a general diffusion matrix $A(x,y)$ satisfying (1.4). The eigenvalue problem (2.14) becomes

$$\left\{\begin{array}{rl} \mu(\lambda',M)\psi^{\lambda',M} = & \nabla\cdot A\nabla\psi^{\lambda',M} + 2\dfrac{\lambda'}{M}\tilde{e}\cdot A\nabla\psi^{\lambda',M} + M\,q\cdot\nabla\psi^{\lambda',M} \\ & +\Big[\Big(\dfrac{\lambda'}{M}\Big)^2\tilde{e}\cdot A\tilde{e} + \lambda'\,q\cdot\tilde{e} + \dfrac{\lambda'}{M}\nabla\cdot A\tilde{e} + \zeta\Big]\psi^{\lambda',M} \text{ in } \Omega, \\ \nu\cdot A\nabla\psi^{\lambda',M} = & -\dfrac{\lambda'}{M}(\nu\cdot A\tilde{e})\psi^{\lambda',M} \text{ on } \partial\Omega \text{ (whenever } \partial\Omega\neq\emptyset). \end{array}\right. \tag{2.29}$$

Similar to the case where $A = Id$, the principal eigenfunctions of (2.29) $\psi^{\lambda',M}$ will be positive in $\overline{\Omega}$, unique up to multiplication by a nonzero constant, and L-periodic with respect to x (see Section 5 in [1]). Consequently, for any sequence $\{M_n\}_n$ in $(0,+\infty)$ such that $M_n \to +\infty$ as $n \to +\infty$, multiplying (respectively dividing) (2.29), for any $\lambda' > 0$ and for $M = M_n$, by ψ^{λ',M_n} and integrating by parts over the cell C implies the boundedness of $\{\nabla\psi^{\lambda',M_n}\}_n$ (respectively $\{\nabla\psi^{\lambda',M_n}/\psi^{\lambda',M_n}\}_n$) in $L^2_{loc}(\Omega)$. On the other hand, the equation (2.15), which was essential in the proof when $A = Id$, will be replaced by

$$\frac{\mu(\lambda',M)}{\lambda'}\int_C w^2 = \frac{D_A(\lambda',M)}{\lambda'} + \int_C (q\cdot\tilde{e})\,w^2 + \frac{1}{\lambda'}\int_C \left[\zeta w^2 - \nabla w\cdot A\nabla w\right], \tag{2.30}$$

for all λ' and M in $(0,+\infty)$, where

$$D_A(\lambda',M) := \int_C \Big(\frac{\nabla\psi^{\lambda',M}}{\psi^{\lambda',M}}\,w - \nabla w + \frac{\lambda'}{M}\tilde{e}\,w\Big)\cdot A\Big(\frac{\nabla\psi^{\lambda',M}}{\psi^{\lambda',M}}\,w - \nabla w + \frac{\lambda'}{M}\tilde{e}\,w\Big).$$

The result of Lemma 2.1 will remain true with (2.17) replaced by

$$\lim_{n\to+\infty} D_A(\lambda',M_n) = 0.$$

Finally, due to the coercivity of the diffusion matrix A given in (1.4), we can easily adapt the proof of the results of Proposition 2.1 to the function

$\lambda \mapsto h^A(\lambda)$ defined by

$$\forall \lambda \in (0,+\infty),\ h^A(\lambda) := \int_C (q\cdot\tilde{e})\, w^2 + \frac{1}{\lambda}\int_C \left[\zeta w^2 - \nabla w\cdot A\nabla w\right], \quad (2.31)$$

which coincides with $\lambda \mapsto h(\lambda)$ defined in (2.4) when $A = Id$.

2.2. **Cases of large advection with small reaction or large diffusion (Proof of Theorem 1.2).** We mention that the proof of (1.12) is very similar to that of (1.11). We are going to prove the limit in (1.11) only.

Step 1. Existence of a maximizer for (1.11). To begin, we prove that $\sup_{w\,\in\,\mathcal{I}} \frac{\int_C (q\cdot\tilde{e})\, w}{\sqrt{\int_C \nabla w\cdot A\nabla w}}$ is finite. For any $w \in \mathcal{I}$, we define

$$\overline{w} := \fint_C w(x)dx,$$

and we write $w = \overline{w} + v$. We notice that $\nabla w \equiv \nabla v$ and, thanks to Poincaré's inequality, we get

$$\|v\|^2_{L^2(C)} \le \kappa \int_C |\nabla v|^2 = \kappa \int_C |\nabla w|^2 \le \frac{\kappa}{\alpha_1}\int_C \nabla w\cdot A\nabla w, \quad (2.32)$$

for some $\kappa > 0$ independent of w and v, where $\alpha_1 > 0$ is given by (1.4). Moreover, it follows from the fourth line in (1.5) that

$$\int_C q\cdot\tilde{e}\, w = \int_C q\cdot\tilde{e}\, v.$$

Thus, applying the Cauchy-Schwarz inequality, we get $\forall w \in \mathcal{I}$,

$$\begin{aligned} \Big|\int_C q\cdot\tilde{e}\, w\Big| = \Big|\int_C q\cdot\tilde{e}\, v\Big| &\le \|q\cdot\tilde{e}\|_{L^2(C)}\|v\|_{L^2(C)} \\ &\le \sqrt{\kappa/\alpha_1}\|q\cdot\tilde{e}\|_{L^2(C)}\sqrt{\int_C \nabla w\cdot A\nabla w}. \end{aligned}$$

Hence, for any $w \in \mathcal{I}$ such that w is not constant,

$$0 \le \frac{\left|\int_C q\cdot\tilde{e}\, w\right|}{\sqrt{\int_C \nabla w\cdot A\nabla w}} \le \sqrt{\frac{\kappa}{\alpha_1}}\|q\cdot\tilde{e}\|_{L^2(C)} < +\infty$$

since $q \in C^{1,\delta}(\overline{\Omega})$. Consequently, the quantity

$$l := \sup_{w\,\in\,\mathcal{I}} \frac{\int_C (q\cdot\tilde{e})\, w}{\sqrt{\int_C \nabla w\cdot A\nabla w}} \ge 0$$

is well defined. We mention that if $\int_C (q \cdot \tilde{e})\, w = 0$ for each $w \in \mathcal{I}$, then $l = 0$ and the supremum is a maximum attained by any nonconstant $w \in \mathcal{I}$. In what follows, we have to treat the case where there exists at least a $w_0 \in \mathcal{I}$ (nonconstant almost everywhere in C) such that $\int_C (q \cdot \tilde{e})\, w_0 \neq 0$ and consequently $l > 0$.

Now, we prove that the above supremum is actually a maximum. We take $\{w_n\}_{n\in\mathbb{N}}$ as a maximizing sequence. As was done in (2.32) above, we may assume that

$$\forall\, n \in \mathbb{N},\ \|w_n\|^2_{L^2(C)} \leq \kappa \int_C |\nabla w_n|^2 \leq \frac{\kappa}{\alpha_1} \int_C \nabla w_n \cdot A \nabla w_n. \tag{2.33}$$

Moreover, we can consider

$$\tilde{w}_n := \frac{w_n}{\sqrt{\int_C \nabla w_n \cdot A \nabla w_n}}$$

as a maximizing sequence. The advantage is that $\{\tilde{w}_n\}_n$ is bounded in $L^2(C)$ (due to (2.33)) and in $H^1(C)$ since $\int_C \nabla \tilde{w}_n \cdot A \nabla \tilde{w}_n = 1$ for each $n \in \mathbb{N}$. As a consequence, there exists $\tilde{w} \in H^1(C)$ such that

$$\tilde{w}_n \to \tilde{w} \ \text{ in } L^2(C) \text{ strongly and } \ \tilde{w}_n \rightharpoonup \tilde{w} \ \text{ in } H^1(C) \text{ weakly as } n \to +\infty.$$

Thus,

$$1 = \liminf_{n\to+\infty} \int_C \nabla \tilde{w}_n \cdot A \nabla \tilde{w}_n \geq \int_C \nabla \tilde{w} \cdot A \nabla \tilde{w}, \tag{2.34}$$

and $\tilde{w} \in \mathcal{I}$ is the weak limit of first integrals of q.

On the other hand, strong convergence in $L^2(C)$, the definition of the maximizing sequence $\{\tilde{w}_n\}_n$, and (2.34) yield that

$$\int_C (q \cdot \tilde{e})\tilde{w} \;=\; \lim_{n\to+\infty} \int_C (q \cdot \tilde{e})\tilde{w}_n = l \geq l \times \int_C \nabla \tilde{w} \cdot A \nabla \tilde{w}. \tag{2.35}$$

We mention that $\tilde{w}$ cannot be constant almost everywhere in C because in this case one gets $l = \int_C (q \cdot \tilde{e})\tilde{w} = \tilde{w} \int_C q \cdot \tilde{e} = 0$, and this contradicts the assumption that $l > 0$.

Therefore, it follows from (2.35) and the definition of l that

$$l = \frac{\int_C (q \cdot \tilde{e})\tilde{w}}{\sqrt{\int_C \nabla \tilde{w} \cdot A \nabla \tilde{w}}},$$

and so the maximum of (1.11) is attained at $\tilde{w}$.

We also obtain $\int_C \nabla \tilde{w} \cdot A \nabla \tilde{w} = 1$, which yields that $\{\tilde{w}_n\}_n$ converges to $\tilde{w}$ in $H^1(C)$ strongly as $n \to +\infty$.

Step 2. Theorem 1.1 yields that, for any $\varepsilon > 0$, the limit of $c^*_{\Omega,A,M\,q,\varepsilon f}(e)/M$ is related to the set

$$\mathcal{I}_1^{A,\varepsilon} := \left\{ w \in \mathcal{I} \text{ such that } \varepsilon \int_C \zeta w^2 \geq \int_C \nabla w \cdot A \nabla w \right\}.$$

As we did at the beginning of Step 1, we write each $w \in \mathcal{I}_1^{A,\varepsilon}$ as $w = v + \overline{w}$, where $\overline{w} = \frac{\int_C w}{|C|}$. For each $w \in \mathcal{I}_1^{A,\varepsilon}$ with $\|w\|_{L^2(C)} = 1$, we have $|\overline{w}| \leq 1/\sqrt{|C|}$. We also have $|\int_C v| \leq \sqrt{|C|}\|v\|_{L^2(C)}$. Owing to (2.32) together with the facts that $w \in \mathcal{I}_1^{A,\varepsilon}$ and ζ is globally bounded, one consequently gets

$$\int_C v = O(\sqrt{\varepsilon}) \text{ as } \varepsilon \to 0^+. \tag{2.36}$$

On the other hand,

$$1 = \int_C w^2 = \overline{w}^2|C| + 2\,\underbrace{\overline{w}\int_C v}_{O(\sqrt{\varepsilon})} + \underbrace{\int_C v^2}_{O(\varepsilon)}.$$

Thus, $\overline{w}^2|C| = 1 + O(\sqrt{\varepsilon})$ as $\varepsilon \to 0^+$. Now, we write

$$\overline{w}^2|C| - 1 = (\overline{w}\sqrt{|C|} - 1)(\overline{w}\sqrt{|C|} + 1) = O(\sqrt{\varepsilon}),$$

and we use the fact that

$$1 \leq \overline{w}\sqrt{|C|} + 1 \leq 2 \text{ when } \overline{w} \geq 0, \text{ and } \quad -2 \leq \overline{w}\sqrt{|C|} - 1 \leq -1 \text{ when } \overline{w} \leq 0,$$

to obtain, $\forall\, w \in \mathcal{I}_1^{A,\varepsilon}$,

$$(\|w\|_{L^2(C)} = 1\ ) \Rightarrow\ \overline{w} = \mathrm{sgn}(\overline{w})\frac{1}{\sqrt{|C|}} + O(\sqrt{\varepsilon}) \quad \text{as} \quad \varepsilon \to 0^+. \tag{2.37}$$

For such w's, having $q \cdot \tilde{e} \in L^\infty(C)$, it then follows from (2.36) and (2.37) that

$$\begin{aligned}
\int_C (q \cdot \tilde{e}) w^2 &= 2\overline{w}\int_C (q \cdot \tilde{e}) v + \int_C (q \cdot \tilde{e}) v^2 \\
&= \frac{2\ \mathrm{sgn}(\overline{w}) \int_C (q \cdot \tilde{e}) v\, dx}{\sqrt{|C|}} + O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0^+ \\
&= \frac{2\ \mathrm{sgn}(\overline{w}) \int_C (q \cdot \tilde{e}) w\, dx}{\sqrt{|C|}} + O(\varepsilon),
\end{aligned}$$

since $\int_C (q\cdot\tilde{e}) = 0$. As $w \in \mathcal{I}_1^{A,\varepsilon}$, we have $\frac{\sqrt{\varepsilon}\sqrt{\int_C \zeta w^2}}{\sqrt{\int_C \nabla w\cdot A\nabla w}} \geq 1$. Hence, $\forall\, w \in \mathcal{I}_1^{A,\varepsilon}$,

$$\int_C (q\cdot\tilde{e})w^2 \leq \frac{\sqrt{\varepsilon}\sqrt{\int_C \zeta w^2}}{\sqrt{\int_C \nabla w \cdot A\nabla w}} \times \frac{2\left|\int_C (q\cdot\tilde{e})w\,dx\right|}{\sqrt{|C|}} + O(\varepsilon), \tag{2.38}$$

whenever $\|w\|_{L^2(C)} = 1$.

Moreover, if $\int_C \nabla w\cdot A\nabla w = \varepsilon \int_C \zeta\, w^2$ and $\|w\|_{L^2(C)} = 1$, then we have

$$\int_C (q\cdot\tilde{e})w^2 = \frac{\sqrt{\varepsilon}\sqrt{\int_C \zeta w^2}}{\sqrt{\int_C \nabla w \cdot A\nabla w}} \times \frac{2\ \mathrm{sgn}(\overline{w})\int_C (q\cdot\tilde{e})w\,dx}{\sqrt{|C|}} + O(\varepsilon). \tag{2.39}$$

Step 3. For a fixed $\varepsilon > 0$, we know from Theorem 1.1 that

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,A,M\,q,\varepsilon f}(e)}{M\sqrt{\varepsilon}} = \max_{\substack{w\in\mathcal{I}_1^{A,\varepsilon},\\ \|w\|_{L^2(C)}=1}} \frac{1}{\sqrt{\varepsilon}}\int_C (q\cdot\tilde{e})w^2 = \frac{1}{\sqrt{\varepsilon}}\int_C (q\cdot\tilde{e})w_\varepsilon^2, \tag{2.40}$$

for some $w_\varepsilon \in \mathcal{I}_1^{A,\varepsilon}$ with $\|w_\varepsilon\|_{L^2(C)} = 1$. However,

$$\|\nabla w_\varepsilon\|^2_{L^2(C)} \leq \frac{1}{\alpha_1}\int_C \nabla w_\varepsilon\cdot A\nabla w_\varepsilon \leq \frac{\varepsilon\|\zeta\|_\infty}{\alpha_1},$$

since $w_\varepsilon \in \mathcal{I}_1^{A,\varepsilon}$. Hence, $\nabla w_\varepsilon \to 0$ in $L^2(C)$ as $\varepsilon \to 0^+$. Consequently, the sequence $\{w_\varepsilon\}_\varepsilon$ is bounded in $H^1(C)$, and thus there exists $w_0 \in H^1(C)$ such that $w_\varepsilon \rightharpoonup w_0$ in $H^1(C)$ weakly, and $w_\varepsilon \to w_0$ in $L^2(C)$ strongly, as $\varepsilon \to 0^+$. Strong convergence in $L^2(C)$ yields that $w_0 \not\equiv 0$ in C and $\|w_0\|_{L^2(C)} = 1$. Besides, weak convergence implies that

$$\int_C |\nabla w_0|^2 \leq \liminf_{\varepsilon\to 0^+} \int_C |\nabla w_\varepsilon|^2 = 0.$$

Therefore, $|\nabla w_0| = 0$ almost everywhere in C and $w_0 = \frac{\pm 1}{\sqrt{|C|}}$ is constant almost everywhere in C. One concludes that

$$\int_C \zeta w_\varepsilon^2 \to \frac{1}{|C|}\int_C \zeta \ \text{ as } \varepsilon \to 0^+.$$

These results together with (2.38) and (2.40) lead to

$$\limsup_{\varepsilon\to 0^+} \lim_{M\to+\infty} \frac{c^*_{\Omega,A,M\,q,\varepsilon f}(e)}{M\sqrt{\varepsilon}} \leq \frac{2\sqrt{\int_C \zeta}}{|C|} \max_{w\,\in\,\mathcal{I}} \frac{\int_C (q\cdot\tilde{e})\,w}{\sqrt{\int_C \nabla w\cdot A\nabla w}}. \tag{2.41}$$

Step 4. To prove equality, we take any maximizer $v \in \mathcal{I}$ (nonconstant) of

$$R(w) := \frac{\int_C (q \cdot \tilde{e})\, w}{\sqrt{\int_C \nabla w \cdot A \nabla w}},$$

over $\mathcal{I}$. It is easy to see that for any $k \in \mathbb{R}$, $v + k$ is also a maximizer of R. Thus, we can choose without any loss of generality v so that $\overline{v} \geq 0$.

We want to prove that there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ we can find a maximizer $w_\varepsilon \in \mathcal{I}$ of R satisfying

$$\|w_\varepsilon\|_{L^2(C)} = 1 \text{ and } \int_C \nabla w_\varepsilon \cdot A \nabla w_\varepsilon = \varepsilon \int_C \zeta\, w_\varepsilon^2. \tag{2.42}$$

Using Poincaré's inequality (as in Step 1), we have

$$\int_C (v - \overline{v})^2 \leq \kappa \int_C |\nabla v|^2$$

for some $\kappa > 0$ depending only on the set C. Moreover, the function ζ is positive and belongs to $L^\infty(C)$. Thus, for any $0 < \varepsilon \leq \varepsilon_0 := \frac{\alpha_1}{\kappa \|\zeta\|_\infty}$ (where α_1 is given by (1.4)), we have

$$\varepsilon \int_C \zeta (v - \overline{v})^2 \leq \int_C \nabla v \cdot A \nabla v. \tag{2.43}$$

Now, let's fix ε in $(0, \varepsilon_0]$. Since $\overline{v} \geq 0$, one can then find a constant $m = m(\varepsilon, v) \leq 0$ depending on ε and v such that

$$\varepsilon \int_C \zeta (v - m)^2 \geq \varepsilon \big(\min_{z \in C} \zeta(z) \big) \int_C (v - m)^2 \geq \int_C \nabla v \cdot A \nabla v. \tag{2.44}$$

The continuity of

$$t \mapsto \varepsilon \int_C (v - t)^2 - \int_C \nabla v \cdot A \nabla v$$

together with (2.43) and (2.44) yield that there exists $r = r(\varepsilon, v) \in [m, \overline{v}]$, such that

$$\varepsilon \int_C \zeta (v - r)^2 = \int_C \nabla v \cdot A \nabla v.$$

We call $w_\varepsilon := \frac{v - r}{||v - r||_{L^2(C)}}$. Then, w_ε satisfies (2.42) and it maximizes $R(w)$ since v does and since $\int_C q \cdot \tilde{e} = 0$. Moreover, $\overline{w}_\varepsilon \geq 0$ since $r \in [m, \overline{v}]$. Imitating the argument used in Step 3, one gets that, up to the extraction of a subsequence, $w_\varepsilon \to \frac{1}{\sqrt{|C|}}$ strongly in $L^2(C)$.

Applying (2.39) for $w = w_\varepsilon$ (where $\mathrm{sgn}(\overline{w}_\varepsilon) = +1$) and since $w_\varepsilon \in \mathcal{I}_1^{A,\varepsilon}$, we then get for any $0 < \varepsilon \leq \varepsilon_0$,

$$\begin{aligned}\max_{w\in\mathcal{I}} R(w) = R(w_\varepsilon) \quad &= \frac{\sqrt{|C|}}{2\sqrt{\varepsilon\int_C \zeta w_\varepsilon^2}}\int_C (q\cdot\tilde{e})w_\varepsilon^2 + O(\sqrt{\varepsilon})\\ &\leq O(\sqrt{\varepsilon}) + \frac{\sqrt{|C|}}{2\sqrt{\int_C \zeta w_\varepsilon^2}} \max_{\substack{w\in\mathcal{I}_1^{A,\varepsilon},\\ \|w\|_{L^2(C)}=1}} \frac{\int_C (q\cdot\tilde{e})w^2}{\sqrt{\varepsilon}}.\end{aligned}$$

In other words, $\forall\, 0 < \varepsilon \leq \varepsilon_0$,

$$\max_{w\in\mathcal{I}} R(w) \leq O(\sqrt{\varepsilon}) + \frac{\sqrt{|C|}}{2\sqrt{\int_C \zeta w_\varepsilon^2}} \lim_{M\to+\infty} \frac{c^*_{\Omega,A,M\,q,\varepsilon f}(e)}{M\sqrt{\varepsilon}}. \tag{2.45}$$

Passing to the limit as $\varepsilon \to 0^+$ in (2.45) and using the strong convergence of w_ε in $L^2(C)$, we obtain

$$\max_{w\in\mathcal{I}} R(w) \leq 0 + \frac{|C|}{2\sqrt{\int_C \zeta}} \liminf_{\varepsilon\to 0^+} \lim_{M\to+\infty} \frac{c^*_{\Omega,A,M\,q,\varepsilon f}(e)}{M\sqrt{\varepsilon}}. \tag{2.46}$$

This inequality and (2.41) finish the proof of (1.11). □

3. THE TWO-DIMENSIONAL CASE ($N = 2$)

In this section, the space dimension is $N = 2$. In what follows, we find the form of any divergence-free advection field and then prove Theorem 1.3 after passing by many auxiliary lemmas. We first start by proving Lemma 1.1, which was announced in Section 1.

Proof of Lemma 1.1. We first prove by contradiction that, for $d = 1$, all the trajectories are L_1e_1-periodic. Indeed, suppose that there exists a periodic unbounded trajectory $T(x)$, which is not L_1e_1-periodic. Then it is pL_1e_1 periodic for some $p \in \mathbb{N}$, $p \geq 2$. By periodicity of q, $T(x)+L_1e_1$ is also an unbounded periodic trajectory of q in Ω, different from $T(x)$. Moreover, $T(x)\cap(T(x)+L_1e_1) = \emptyset$, because two different trajectories never intersect.

We set

$$\begin{aligned} m &:= \min\{y_2 \text{ such that } (y_1,y_2)\in T(x)\},\\ M &:= \max\{y_2 \text{ such that } (y_1,y_2)\in T(x)\}.\end{aligned}$$

We have $m \neq M$; otherwise, $T(x)$ is a horizontal straight line and is L_1e_1-periodic.

We also define

$$a_1 := \min\{y_1 \geq 0 \text{ such that } (y_1, m) \in T(x)\},$$
$$a_2 := \min\{y_1 \geq a_1 \text{ such that } (y_1, M) \in T(x)\}.$$

Let $T'(x) := T(x) \setminus (\{(a_1, m)\} \cup \{(a_2, M)\})$. Since $T(x)$ is a simple curve, $T'(x)$ has exactly three connected components, two of which are unbounded. Let $T_b(x)$ be the bounded component of $T'(x)$; we set $T_a(x) = \overline{T_b(x)}$, which is a compact subset of $T(x)$, with boundary $\{(a_1, m)\} \cup \{(a_2, M)\}$. We define then

$$b_1 = \min\{y_1 \text{ such that } (y_1, y_2) \in T_a(x)\},$$
$$b_2 = \max\{y_1 \text{ such that } (y_1, y_2) \in T_a(x)\}.$$

We define the following curve:

$$\mathcal{C} := T_a(x) \cup \{(a_1, y_2),\ y_2 < m\} \cup \{(a_2, y_2),\ y_2 > M\}.$$

The curve $\mathcal{C}$ is a simple connected curve, which splits Ω into several connected components, two of which are unbounded. Let Ω_1 be the left unbounded component, more precisely the component containing the set $\{y = (y_1, y_2) \in \Omega$ such that $y_1 < b_1\}$, and Ω_2 be the right unbounded component, the one containing the set $\{y = (y_1, y_2) \in \Omega$ such that $y_1 > b_2\}$.

Let $z \in \Omega_1$ and $z' \in \Omega_2$ be two points of $T(x) + L_1 e_1$; then, following $T(x) + L_1 e_1$, there is a continuous path in Ω from z to z'. This path must cross $\mathcal{C}$ for obvious reasons of continuity. However, it can not cross $\{(a_1, y_2) : y_2 < m\}$ or $\{(a_2, y_2) : y_2 > M\}$ because of the definition of m and M.

Hence this path, which is a subset of $T(x) + L_1 e_1$, must cross $T_a(x)$, which is a subset of $T(x)$. This leads to $T(x) \cap (T(x) + L_1 e_1) \neq \emptyset$. This is a contradiction and proves the lemma for $d = 1$.

In the case $d = 2$, we need to prove that if there exists an $\mathbf{a}$-periodic unbounded trajectory, then any other unbounded periodic trajectory will be $\mathbf{a}$-periodic. The idea is to reduce this proof to the proof of the case $d = 1$. Suppose that there exists an $\mathbf{a}$-periodic unbounded trajectory $T(x)$ of q in Ω. We set $e_1' = \mathbf{a}/|\mathbf{a}|$, and e_2' such that (e_1', e_2') is a direct orthonormal frame of $\mathbb{R}^2$. In this new basis, q is L_1'-periodic in the e_1' directions, with $L_1' = |\mathbf{a}|$.

Thus, $T(x)$ is bounded in the e_2' direction. Suppose now that $y = y_1' e_1' + y_2' e_2' \in \Omega$ is such that $T(y)$ is an unbounded periodic trajectory. Let $z_1 \in L_1\mathbb{Z} \times L_2\mathbb{Z}$ such that $\inf_{z \in T(x)+z_1} z \cdot e_2' > y_2'$, and $z_2 \in L_1\mathbb{Z} \times L_2\mathbb{Z}$ such that $\sup_{z \in T(x)+z_2} z \cdot e_2' < y_2'$. We have that $T(x) + z_1$ and $T(x) + z_2$ are two trajectories of q in Ω, and they split Ω into three connected components, one of which is bounded in the e_2' direction. We shall denote it Ω_b. By construction $y \in \Omega_b$. Since two different trajectories can not intersect, $T(y)$ must stay inside Ω_b, and using the same procedure as the case $d = 1$ we conclude that $T(y)$ is $L_1' e_1' = \mathbf{a}$-periodic. □

Proposition 3.1. *Let $d = 1$ or 2, where d is defined in (1.2). Let $q \in C^{1,\delta}(\overline{\Omega})$, L-periodic with respect to x and satisfying the conditions*

$$\left\{\begin{array}{rcl} \int_C q & = & 0, \\ \nabla \cdot q & = & 0 \text{ in } \Omega, \\ q \cdot \nu & = & 0 \text{ on } \partial\Omega. \end{array}\right. \tag{3.1}$$

Then, there exists $\phi \in C^{2,\delta}(\overline{\Omega})$, L-periodic with respect to x, such that

$$q = \nabla^{\perp}\phi \quad \text{in } \Omega. \tag{3.2}$$

Moreover, ϕ is constant on every connected component of $\partial\Omega$.

Remark 3.1. We mention that the representation $q = \nabla^{\perp}\phi$ is already known in the case where the domain Ω is bounded and **simply connected** or equal to the whole space $\mathbb{R}^2$. However, the above proposition applies in more cases due to the condition $q \cdot \nu = 0$ on $\partial\Omega$ (see the proof below). For example, it applies when Ω is the whole space $\mathbb{R}^2$ with a periodic array of holes or when Ω is an infinite cylinder which may have an oscillating boundary and/or a periodic array of holes.

Proof of Proposition 3.1. We first consider the case where $d = 2$. We define $\hat{\Omega} := \Omega/(L_1\mathbb{Z} \times L_2\mathbb{Z})$ and $T := \mathbb{R}^2/(L_1\mathbb{Z} \times L_2\mathbb{Z})$. If $x \in \mathbb{R}^2$, we denote by $\hat{x}$ its class of equivalence in T, and if $\phi : \mathbb{R}^2 \to \mathbb{R}$ is L-periodic, we denote by $\hat{\phi}$ the function $T \to \mathbb{R}^2$ satisfying $\phi(x) = \hat{\phi}(\hat{x})$.

Finding $\phi \in C^{2,\delta}(\overline{\Omega})$, which is L-periodic with respect to x and satisfying (3.2), is then equivalent to finding $\hat{\phi} \in C^{2,\delta}(\overline{\hat{\Omega}})$ satisfying (3.2). We consider the map $\tilde{q}$ defined as follows:

$$\tilde{q} : T \longrightarrow \mathbb{R}^2, \quad \hat{x} \in \overline{\hat{\Omega}} \longmapsto q(x), \quad \hat{x} \notin \overline{\hat{\Omega}} \longmapsto 0.$$

We claim that $\tilde{q}$ is a divergence-free vector field on T in the sense of distributions. Indeed, if $\psi \in C^{\infty}(T)$, we then have

$$\begin{array}{rcl} \langle \operatorname{div}(\tilde{q}), \psi \rangle & := & -\langle \tilde{q}, \nabla\psi \rangle \;=\; -\displaystyle\int_T \tilde{q} \cdot \nabla\psi \\ & = & -\displaystyle\int_{\hat{\Omega}} q \cdot \nabla\psi = -\int_{\partial\hat{\Omega}} \psi\, q \cdot \nu + \int_{\hat{\Omega}} \psi \nabla \cdot q = 0 + 0 = 0, \end{array}$$

because of the conditions (3.1). Moreover, we clearly have $\int_T \tilde{q} = 0$. Now, we denote by R the matrix of a direct rotation of angle $\pi/2$.

The next step is to solve the following equation in $H^1(T)$:

$$-\Delta\tilde{\phi} = \nabla\cdot(R\tilde{q}). \tag{3.3}$$

A function $\tilde{\phi} \in H^1(T)$ is a weak solution of (3.3) if we have, for all $\psi \in H^1(T)$,

$$\int_T \nabla\tilde{\phi}\cdot\nabla\psi = -\int_T R\tilde{q}\cdot\nabla\psi. \tag{3.4}$$

We set $E := \{\psi \in H^1(T) \text{ such that } \int_T \psi = 0\}$, so that, thanks to Poincaré's inequality,

$$\langle u, v\rangle_E := \int_T \nabla u\cdot\nabla v$$

is an inner product on E. Moreover, $\psi \in E \mapsto \int_T R\tilde{q}\cdot\nabla\psi$ is a continuous linear form on E, so by the Lax-Milgram theorem, there exists a unique $\tilde{\phi} \in E$ solution of (3.4). The condition $\int_T \psi = 0$ is not restrictive because only the gradients of functions belonging to E appear in the weak formulation (3.4). We then have $\tilde{\phi} \in H^1(T)$ such that in the sense of distributions

$$\begin{aligned}&\nabla\cdot R(\tilde{q} - \nabla^{\perp}\tilde{\phi}) = 0 \text{ in } T \quad \text{and}\\ &\quad \nabla\cdot(\tilde{q} - \nabla^{\perp}\tilde{\phi}) = 0 \text{ in } T \;\; \text{since } \nabla\cdot\tilde{q} = 0 \text{ in } \mathcal{D}'(T) \text{ and } \operatorname{div}(\nabla^{\perp}\cdot) = 0.\end{aligned}$$

This implies that $\tilde{q} - \nabla^{\perp}\tilde{\phi}$ is a harmonic distribution on T. Using Weyl's theorem (see [10]), we conclude that $\tilde{q} - \nabla^{\perp}\tilde{\phi}$ is a harmonic function on the torus T and therefore is constant. Indeed, if h is a harmonic scalar function on T, then by multiplying h by Δh and integrating by parts we get $\int_T |\nabla h|^2 = 0$, which leads to h being constant on T.

Finally, since $\int_T(\tilde{q} - \nabla^{\perp}\tilde{\phi}) = 0$ and $\tilde{q} - \nabla^{\perp}\tilde{\phi}$ is constant, we conclude that, in the sense of distributions,

$$\tilde{q} = \nabla^{\perp}\tilde{\phi}. \tag{3.5}$$

We set $\hat{\phi} := \tilde{\phi}|_{\hat{\Omega}}$, which solves (3.2) in $\hat{\Omega}$. The corresponding L-periodic function $\phi \in H^1_{loc}(\Omega)$ solves then (3.2) in Ω. The $C^{2,\delta}$ regularity of ϕ in $\overline{\Omega}$ is a consequence of the Schauder estimates for the Laplace equation.

The fact that ϕ is constant on every connected component of $\partial\Omega$ is a straightforward consequence of the identity

$$\nabla^{\perp}\phi\cdot\nu = q\cdot\nu = 0 \;\; \text{on } \partial\Omega.$$

For the case $d = 1$, we symmetrize the set Ω (resp. the cell C) and the field q with respect to the line $y = R$ and we call the resulting set by Ω_s

(resp. C_s) and the resulting vector field by q_s. For the sake of completeness, we mention that $q_s(x,y)$ is given by

$$q_s(x,y)=\begin{cases} q(x,y) \text{ for } (x,y)\in\Omega,\\ q_1(x,2R-y)e_1-q_2(x,2R-y)e_2 \text{ for } (x,y)\in\Omega_s\setminus\Omega.\end{cases}$$

One can easily notice that $\int_{C_s} q_s=0$. We generate (in the direction e_2) a periodic set Ω_1 from the set Ω in order to reduce this case to the case $d=2$. For that purpose, since we already have $\Omega_s\subset\mathbb{R}\times[-R,3R]$ (take $d=1$ and $N=2$ in (1.2)), we define Ω_1 in the following way:

$$\Omega_1:=\bigcup_{i\in\mathbb{Z}}\{\Omega_s+i(4R+2)e_2\}. \tag{3.6}$$

Thus, Ω_1 is periodic in the direction of e_1 and e_2 and is the disjoint reunion of translations of Ω_s. We set

$$\hat{\Omega}:=\Omega_1/(L_1\mathbb{Z}\times(4R+2)\mathbb{Z}) \ \text{ and } \ T:=\mathbb{R}^2/(L_1\mathbb{Z}\times(4R+2)\mathbb{Z}),$$

and the procedure used for the case $d=2$ still works in this case and gives $\phi\in C^{2,\delta}(\overline{\Omega})$, L-periodic, solving $q=\nabla^\perp\phi$. □

Corollary 3.1. *Let*

$$\mathcal{J}:=\{\eta\circ\phi \text{ such that } \eta:\mathbb{R}\to\mathbb{R} \text{ is Lipschitz}\}, \tag{3.7}$$

where ϕ, such that $q=\nabla^\perp\phi$, is given by Proposition 3.1. *Then, $\mathcal{J}\subset\mathcal{I}\cup\{0\}$.*

Proof. We first mention that if $q\not\equiv 0$ is the advection vector field and ϕ is the function given by Proposition 3.1, then $\phi\in\mathcal{I}$. Indeed, $\phi\in H^1_{loc}(\Omega)$ and we have

$$\forall\, z\in\Omega,\ q(z)\cdot\nabla\phi(z)=\nabla^\perp\phi(z)\cdot\nabla\phi(z)=0.$$

Now, using Remark 1.1, we conclude that $\eta\circ\phi$ is a first integral whenever $\eta:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function. □

Remark 3.2. For any $w\in\mathcal{J}$, we have $\int_C(q\cdot\tilde{e})w^2=0$. Indeed, $w=\eta\circ\phi$ and $q=\nabla^\perp\phi$, which gives

$$\int_C(q\cdot\tilde{e})w^2=\tilde{e}\cdot\int_C\left(\nabla^\perp\phi\right)\,\eta^2(\phi)=\tilde{e}\cdot R\int_C\nabla\left(F\circ\phi\right),$$

where R the matrix of a direct rotation of angle $\pi/2$ and $F'=\eta^2$. Let $\tilde{\phi}\in H^1(T)$ be defined by (3.5); then $\tilde{\phi}=\phi$ on $\hat{\Omega}$ and is constant on every connected component of $T\backslash\hat{\Omega}$, and so is $F\circ\tilde{\phi}$. We then have

$$\int_{T\backslash\hat{\Omega}}\nabla\left(F\circ\tilde{\phi}\right)=0.$$

Hence,

$$\int_C (q\cdot\tilde{e})w^2 \;=\; \tilde{e}\cdot R\int_T \nabla\left(F\circ\tilde{\phi}\right)=0,$$

because T has no boundary. Thus,

$$\forall\, w\in\mathcal{J},\ \int_C (q\cdot\tilde{e})w^2=0. \tag{3.8}$$

We recall that the family of first integrals $\mathcal{I}$ always contains the set $\mathcal{J}$. However, this does not, in general, provide enough information about the following quantities,

$$\sup_{w\in\mathcal{I}}\int_C (q\cdot\tilde{e})w^2 \text{ or } \max_{w\in\mathcal{I}_1}\int_C (q\cdot\tilde{e})w^2,$$

which appear in the asymptotics of the minimal speed within a large drift. Lemmas 3.1 and 3.2 are devoted to proving Theorem 1.3 and treat this situation.

Definition 3.1. *Throughout the rest of this section, we denote*

$$\begin{cases} T \;:=\; \mathbb{R}^2/(L_1\mathbb{Z}\times L_2\mathbb{Z}) \quad \textit{and} \quad \hat{\Omega} \;:=\; \Omega/(L_1\mathbb{Z}\times L_2\mathbb{Z}) \quad \textit{if} \quad d=2,\\ T \;:=\; \mathbb{R}^2/\left(L_1\mathbb{Z}\times\{0\}\right) \quad \textit{and} \quad \hat{\Omega} \;:=\; \Omega/\left(L_1\mathbb{Z}\times\{0\}\right) \quad \textit{if} \quad d=1. \end{cases} \tag{3.9}$$

Moreover, if $x\in\Omega$ (respectively $\mathbb{R}^2$), $\hat{x}$ denotes its class of equivalence in $\hat{\Omega}$ (respectively T), and if $u:\Omega\to\mathbb{R}$ is L-periodic, $\hat{u}$ denotes the function $\hat{\Omega}\to\mathbb{R}$ satisfying $\hat{u}(\hat{x})=u(x)$ for almost every $x\in\Omega$.

We also define the canonical projection on T by

$$\Pi:\mathbb{R}^2\longrightarrow T,\quad x\longmapsto\hat{x}. \tag{3.10}$$

We need the following preliminary lemma in order to prove the main theorem of this section:

Lemma 3.1. *Let $\hat{\Omega}$ be the set defined in* (3.6), *$\hat{U}$ be an open subset of $\hat{\Omega}$, and $\hat{\phi}$ be given by* (3.2). *We suppose the following:*

(i) $\hat{q}(\hat{x})\neq 0$ for all $\hat{x}\in\hat{U}$,

(ii) the level sets of $\hat{\phi}$ in $\hat{U}$ are all connected.

Then, for every $w\in\mathcal{I}$, there exists a continuous function $\eta:\hat{\phi}(\hat{U})\to\mathbb{R}$ such that

$$\hat{w}=\eta\circ\hat{\phi} \text{ on } \hat{U}. \tag{3.11}$$

Proof. For every $\lambda\in\hat{\phi}(\hat{U})$, we denote by Γ_λ the level set

$$\Gamma_\lambda:=\{\hat{x}\in\hat{U} \text{ such that } \hat{\phi}(x)=\lambda\}.$$

It follows from (i) that $\nabla\hat{\phi}$ does not vanish on $\hat{U}$, and hence $\hat{w}$ has to be constant on the connected components of the level sets of $\hat{\phi}$ because

$$\forall \hat{x} \in \hat{U},\ \nabla^{\perp}\hat{\phi}(\hat{x}) \cdot \nabla \hat{w}(\hat{x}) = q(x) \cdot \nabla w(x) = 0.$$

By (ii), the level sets of $\hat{\phi}$ in $\hat{U}$ are connected, so $\hat{w}$ is constant on every level set of $\hat{\phi}$. If $\hat{x} \in \Gamma_\lambda$, we have $\hat{\phi}(\hat{x}) = \lambda$, and $\hat{w}$ is constant on Γ_λ, and so depends only on λ. Then, we can define η by $\eta(\lambda) = \hat{w}(\Gamma_\lambda)$. To prove the continuity of η, we suppose, to the contrary, that there exists $\lambda_0 \in \hat{\phi}(\hat{U})$ such that η is not continuous at λ_0. The set Γ_{λ_0} is a curve because $\hat{U}$ is open and $\nabla\hat{\phi}$ does not vanish on $\hat{U}$ by (i). The function $\hat{w}$ then has a "jump" along the level set Γ_{λ_0}, which is impossible because $\hat{w} \in H^1(\hat{U})$, and so has a trace on Γ_{λ_0}. $\square$

Recalling Definition 1.3 of the trajectories of an advection field q, we mention the following:

Remark 3.3. It is obvious that ϕ is constant on every trajectory of q. Moreover, the trajectories of q make a partition of $\Omega\backslash\{x \in \Omega$ such that $q(x) = 0\}$.
If $T(x)$ is the trajectory of q in Ω, and $T(\hat{x})$ is the trajectory of $\hat{q}$ in $\hat{\Omega}$, then we have

$$T(\hat{x}) = \Pi(T(x)).$$

In other words, the trajectory of the projection is the projection of the trajectory.

Definition 3.2. *We define here the set of "regular trajectories" in $\hat{\Omega}$. Let*

$$\hat{U} := \left\{\hat{x} \in \hat{\Omega} \text{ such that } T(\hat{x}) \text{ is well defined and closed in } \overline{\hat{\Omega}}\right\}. \tag{3.12}$$

We denote by $\hat{U}_i$ the connected components of $\hat{U}$.

The set $\hat{U}$ is exactly the union of the trajectories which are simple closed curves in $\hat{\Omega}$. This is proved in the following proposition:

Proposition 3.2. *Let $\hat{x} \in \hat{U}$; then $T(\hat{x})$ is a C^1 simple closed curve in $\overline{\hat{\Omega}}$.*

Proof. Let $\hat{x} \in \hat{U}$. By definition of $\hat{U}$, $T(\hat{x})$ is closed. Moreover, since $T(\hat{x})$ is a subset of $\overline{\hat{\Omega}}$ which is a compact set, $T(\hat{x})$ is then compact. $\hat{x} \mapsto |\hat{q}(\hat{x})|$ attains then its minimum on $T(\hat{x})$ at some point $\hat{x}_0 \in T(\hat{x})$. Since $\hat{x}_0 \in T(\hat{x})$ we have $|\hat{q}(\hat{x}_0)| = \eta > 0$. We then get

$$\min_{\hat{y} \in T(\hat{x})} |\hat{q}(\hat{x})| \geq \eta > 0. \tag{3.13}$$

Besides, we know that $\hat{\phi}$ is constant on $T(\hat{x})$. Let $\alpha := \hat{\phi}(T(\hat{x}))$, and $A_\alpha := \{\hat{x} \in \hat{\Omega}$ such that $\phi(x) > \alpha\}$. Since $\nabla\hat{\phi}$ does not vanish on $T(\hat{x})$ because of (3.13), we get $T(\hat{x}) \subset \partial A_\alpha$. Using the Stokes formula on A_α with the vector field $\nabla\hat{\phi}$ gives

$$\int_{A_\alpha} -\Delta\hat{\phi} = \int_{\partial A_\alpha} |\nabla\hat{\phi}|.$$

Thus,

$$\eta\mathcal{L}^1(T(\hat{x})) \leq \int_{T(\hat{x})} |\nabla\hat{\phi}| \leq \int_{\hat{\Omega}} |\Delta\hat{\phi}| < +\infty,$$

where $\mathcal{L}^1$ denote the 1-dimensional Lebesgue measure. The trajectory $T(\hat{x})$ is then a C^1 curve with finite length. It has no self intersection point because such a point would be a critical point of $\hat{\phi}$. $T(\hat{x})$ has no boundary because, if $\hat{y} \in \partial T(\hat{x})$, since $T(\hat{x})$ is closed, $\hat{y} \in T(\hat{x})$, and so $\hat{q}(\hat{y}) \neq 0$ and we could extend $T(\hat{x})$ at the point $\hat{y}$. Consequently, if $\hat{x} \in \hat{U}$, $T(\hat{x})$ is a C^1 simple closed curve with finite length. □

Lemma 3.2. *Let $\hat{U}_i$ be as in the previous definition. Then,*

(i) all the level sets of $\hat{\phi}$ in $\hat{U}_i$ are connected,

(ii) all the level sets of $\hat{\phi}$ in $\hat{U}_i$ are homeomorphic,

(iii) $\partial\hat{U}_i$ has exactly two connected components $\hat{\gamma}_1$ and $\hat{\gamma}_2$ such that

$$\hat{\phi}(\hat{\gamma}_1) = \sup_{\hat{x}\in\hat{U}_i} \hat{\phi}(\hat{x}) \text{ and } \hat{\phi}(\hat{\gamma}_2) = \inf_{\hat{x}\in\hat{U}_i} \hat{\phi}(\hat{x}).$$

Proof. Let $\Gamma_\lambda := \{\hat{x} \in \hat{U}_i$ such that $\hat{\phi}(\hat{x}) = \lambda\}$ be a nonempty level set of $\hat{\phi}$ in $\hat{U}_i$. Let Γ_λ^1 be one of its connected components. Γ_λ^1 is a C^1 curve because $\hat{\phi} \in C^{2,\delta}$ and $\nabla\hat{\phi}$ does not vanish on Γ_λ^1 by definition of $\hat{U}$. Let $\hat{x} \in \Gamma_\lambda^1$; we consider the following ODE:

$$y'(t) = \frac{\nabla\hat{\phi}(y(t))}{|\nabla\hat{\phi}(y(t))|^2}, \qquad y(0) = \hat{x}. \tag{3.14}$$

By classical ODE theory, there exists a maximal interval $(t_{\hat{x}}^1, t_{\hat{x}}^2)$ with $t_{\hat{x}}^1 < 0 < t_{\hat{x}}^2$ on which there exists a C^1 solution $y_{\hat{x}}$ of (3.14). Moreover, either $y(t_{\hat{x}}^1)$ (respectively $y(t_{\hat{x}}^2)$) is a critical point of $\hat{\phi}$ or belongs to $\partial\hat{\Omega}$; otherwise, we could extend the maximal solution $y_{\hat{x}}$ to a larger interval, which is impossible. We set

$$t^1 := \max_{\hat{x}\in\Gamma_\lambda^1} t_{\hat{x}}^1 \quad \text{and} \quad t^2 := \min_{\hat{x}\in\Gamma_\lambda^1} t_{\hat{x}}^2.$$

Since t^1 (respectively t^2) is equal to $t_{\hat{x}}^1$ (respectively $t_{\hat{x}}^2$) for some point $\hat{x} \in \Gamma_\lambda^1$, we then have $t_1 < 0 < t_2$. We define the function g on $(t_1, t_2) \times \Gamma_\lambda^1$ by

$g_t(\hat{x}) = y_{\hat{x}}(t)$. We claim that $\phi(g_t(\hat{x})) = \lambda + t$ for every $(t, \hat{x}) \in (t_1, t_2) \times \Gamma^1_\lambda$. Indeed, we have

$$\begin{aligned}\frac{d}{dt}\hat{\phi}(g_t(\hat{x})) &= \frac{d}{dt}\hat{\phi}(y_{\hat{x}}(t)) = y'_{\hat{x}}(t) \cdot \nabla\hat{\phi}(y_{\hat{x}}(t)) \\ &= \frac{\nabla\hat{\phi}(y_{\hat{x}}(t))}{|\nabla\hat{\phi}(y_{\hat{x}}(t))|^2} \cdot \nabla\hat{\phi}(y_{\hat{x}}(t)) = 1,\end{aligned}$$

which leads to $\hat{\phi}(g_t(\hat{x})) - \hat{\phi}(\hat{x}) = t = \hat{\phi}(g_t(\hat{x})) - \lambda$. Thus, on $g_t(\Gamma^1_\lambda)$, $\hat{\phi}$ is equal to the constant $\lambda + t$. Moreover, g_t is continuous with respect to $\hat{x}$ due to the continuity of the solution of an ODE with respect to initial data. We set now

$$\hat{V}_i := \bigcup_{t_1 < t < t_2} g_t(\Gamma^1_\lambda).$$

We need the following two claims, whose proofs are postponed until the end, in order to prove that $\hat{V}_i = \hat{U}_i$:

Claim 1: For every $\varepsilon > 0$, there exists r_ε such that for every $\hat{x} \in \hat{V}_i$,

$$\hat{x} \in \bigcup_{t_1+\varepsilon \le t \le t_2-\varepsilon} g_t(\Gamma^1_\lambda) \Rightarrow B(\hat{x}, r_\varepsilon) \subset \hat{V}_i,$$

and as a consequence $\hat{V}_i$ is an open subset of T.

Claim 2: $\partial\hat{V}_i$ has exactly two connected components C_1 and C_2 such that $\hat{\phi}|_{C_1} \equiv \lambda + t_1$ and $\hat{\phi}|_{C_2} \equiv \lambda + t_2$, and either C_1 (respectively C_2) is a connected component of $\partial\hat{\Omega}$ or contains a critical point of $\hat{\phi}$.

By definition, $\hat{V}_i$ is the union of connected components of level sets of $\hat{\phi}$ on which $\nabla\hat{\phi}$ does not vanish. Hence, $\hat{V}_i \subset \hat{U}$. Moreover, $\hat{V}_i$ is connected because, by construction, any point $\hat{x} \in \hat{V}_i$ can be connected to Γ^1_λ, which is a connected set. Finally, $\hat{U}_i \cap \hat{V}_i \neq \emptyset$ because it contains Γ^1_λ. We then can affirm that $\hat{V}_i \subset \hat{U}_i$.

Suppose that this inclusion is strict; then we can find $\hat{x}_0 \in \hat{U}_i \backslash \hat{V}_i$. Let $\hat{x}_1 \in \hat{V}_i$, and $\gamma : [0, 1] \to \hat{U}_i$ be a continuous path connecting $\hat{x}_0$ and $\hat{x}_1$. By continuity, it crosses $\partial\hat{V}_i$. However, $\partial\hat{V}_i \cap \hat{U}_i = \emptyset$ because, by Claim 2, the connected components of $\partial\hat{V}_i$ are either connected components of $\partial\hat{\Omega}$, which do not intersect $\hat{U}_i$, or contain a critical point of $\hat{\phi}$, and are thus removed from $\hat{U}$ by construction.

Properties (i) and (ii) are straightforward, because a level set of $\hat{U}_i$ is a level set of $\hat{V}_i$ and can be written in the form $g_t(\Gamma^1_\lambda)$, and g_t is a homeomorphism for any $t_1 < t < t_2$. The level sets are then all homeomorphic to Γ^1_λ, which is connected.

Property (iii) has already been proved for $\hat{V}_i$, and $\hat{V}_i = \hat{U}_i$. Eventually, the proof of the lemma is complete. □

Proof of Claim 1. We prove this claim by contradiction. First, we have

$$\bigcup_{t_1+\varepsilon \le t \le t_2-\varepsilon} g_t(\Gamma_\lambda^1) = g([t_1+\varepsilon, t_2-\varepsilon], \Gamma_\lambda^1),$$

where we denote $g(t,\hat{x}) = g_t(\hat{x})$. Moreover, g is a continuous function, because of the continuity of the solution of an ODE with respect to the initial conditions. Therefore, $\bigcup_{t_1+\varepsilon \le t \le t_2-\varepsilon} g_t(\Gamma_\lambda^1)$ is compact, as it is the image of a compact set by a continuous mapping.

Suppose now that Claim 1 is not true. Then, there exist $\varepsilon > 0$ and $\hat{x}$ such that $\hat{x} \in \partial\hat{V}_i$ and $\hat{\phi}(\hat{x}) =: \lambda + \alpha \in [\inf_{\hat{V}_i} \hat{\phi} + \varepsilon, \sup_{\hat{V}_i} \hat{\phi} - \varepsilon]$.

Let ψ defined on $(-\beta, \beta)$ be an arc length local parametrization of $g_\alpha(\Gamma_\lambda^1)$ such that $\psi(0) = \hat{x}$. We define the following mapping for $0 < \xi < \varepsilon$:

$$G : (-\beta,\beta) \times (-\xi,\xi) \longrightarrow \hat{V}_i, \quad (s,t) \longmapsto g_t(\psi(s)).$$

We have $G(0,0) = \hat{x}$ and

$$DG(0,0)(s,t) = s\psi'(0) + t\frac{\nabla\hat{\phi}(\hat{x})}{|\nabla\hat{\phi}(\hat{x})|^2}.$$

The linear mapping $DG(0,0)$ is then an automorphism of $\mathbb{R}^2$, because $\{\psi'(0), \frac{\nabla\hat{\phi}(\hat{x})}{|\nabla\hat{\phi}(\hat{x})|^2}\}$ is an orthogonal basis of $\mathbb{R}^2$. The application of the inverse mapping theorem then gives two open sets $W_1 \subset (-\beta,\beta) \times (-\xi,\xi)$, with $(0,0) \in W_1$, and $W_2 \subset \hat{V}_i$, with $\hat{x} \in W_2$, such that G is a local diffeomorphism from W_1 to W_2. This prevents $\hat{x}$ from belonging to $\partial\hat{V}_i$ and then gives a contradiction.

The fact that $\hat{V}_i$ is an open subset of T is a straightforward consequence, because if $\hat{x} \in \hat{V}_i$, then for ε small enough $\hat{x} \in \bigcup_{t_1+\varepsilon \le t \le t_2-\varepsilon} g_t(\Gamma_\lambda^1)$ and so we can find a neighborhood of $\hat{x}$ in $\hat{V}_i$.

Proof of Claim 2. Let $\hat{x} \in \partial\hat{V}_i$; then $\hat{\phi}(\hat{x}) = \lambda + t_1$ or $\lambda + t_2$. Indeed, by continuity of $\hat{\phi}$ we have $\hat{\phi}(\hat{x}) \in [t_1, t_2]$. Suppose for the sake of contradiction that $\hat{\phi}(\hat{x}) = \alpha \in (t_1, t_2)$; then for ε sufficiently small we have $\alpha \in [t_1+2\varepsilon, t_2-2\varepsilon]$. Let $\{\hat{x}_p\}$ be a sequence in $\hat{V}_i$ converging to $\hat{x}$. We have then $\hat{\phi}(\hat{x}_p) \to \alpha$ as $p \to \infty$. Hence for p large enough we have $\hat{\phi}(\hat{x}_p) \in [t_1+\varepsilon, t_2-\varepsilon]$, so by Claim 1, there exists $r > 0$ such that for p large enough $\mathrm{dist}(\hat{x}_p, \partial\hat{V}_i) \ge r > 0$, leading to $\mathrm{dist}(\hat{x}, \partial\hat{V}_i) \ge r > 0$, which contradicts the fact that $\hat{x} \in \partial\hat{V}_i$.

We now prove that $\partial\hat{V}_i$ has exactly two connected components. Let $C_1 := \partial\hat{V}_i \cap \hat{\phi}^{-1}(\lambda + t_1)$ and $C_2 := \partial\hat{V}_i \cap \hat{\phi}^{-1}(\lambda + t_2)$. We have, using the previous remark,

$$C_1 = \bigcap_{p\in\mathbb{N}} \overline{\hat{V}_i \cap \hat{\phi}^{-1}((\lambda + t_1, \lambda + t_1 + 1/p))}.$$

Since $\hat{V}_i \cap \hat{\phi}^{-1}((\lambda + t_1, \lambda + t_1 + 1/p))$ is a nonempty bounded open connected subset of T, its closure is a nonempty compact connected subset of T. Therefore, C_1 is the decreasing intersection of nonempty connected compact subsets of T, and is then a connected nonempty compact subset of T. Similarly, C_2 is connected.

Finally, we prove that C_1 (respectively C_2) is either a connected component of $\partial\hat{\Omega}$ or contains a critical point of $\hat{\phi}$. We suppose then that C_1 does not contain any critical point of $\hat{\phi}$. It must then contain a point $\hat{x}_0$ of $\partial\hat{\Omega}$; otherwise, for any $\hat{x} \in \Gamma^1_\lambda$ the solution of (3.14) could be extended at the point t_1, and this would contradict the definition of t_1. We denote by D_1 the connected component of $\partial\hat{\Omega}$ containing $\hat{x}_0$. We are left to prove that $D_1 = C_1$. First, we know that C_1 is a regular simple closed curve, because it is a connected component of a level set of $\hat{\phi}$ on which $\hat{\phi}$ does not vanish. We have then $C_1 = T(\hat{x}_0)$. Moreover, the trajectories of $\hat{q}$ intersecting the boundary of $\hat{\Omega}$ follow the boundary of $\hat{\Omega}$ since $q\cdot\nu = 0$ on $\partial\Omega$, so $T(\hat{x}_0) \subset D_1$. We conclude that $C_1 \subset D_1$, and since C_1 and D_1 are both connected simple closed curves we get $C_1 = D_1$. Similarly, we get that C_2 contains a critical point of $\hat{\phi}$ or is a connected component of $\partial\hat{\Omega}$. $\square$

Proof of Theorem 1.3. Using (3.2), we have for any $w \in \mathcal{I}$

$$\int_C qw^2 = R\int_C (\nabla\phi)w^2 = R\int_{\hat{\Omega}} (\nabla\hat{\phi})\hat{w}^2.$$

Let $W := \{\hat{x} \in \hat{\Omega}$ such that $\hat{\phi}(\hat{x})$ is a critical value of $\hat{\phi}\}$. Using the co-area formula ([7], [8]) we get

$$\Big|\int_W \hat{w}^2\nabla\hat{\phi}\Big| \le \int_W \hat{w}^2|\nabla\hat{\phi}| = \int_{\hat{\phi}(W)} \Big(\int_{\hat{\phi}^{-1}(t)} \hat{w}^2(x)\Big)dt.$$

Moreover, from Sard's theorem (see [13], e.g.), since $\hat{\phi}$ is C^2, $\mathcal{L}^1(\hat{\phi}(W)) = 0$, where $\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$. It follows that

$$\int_W \hat{w}^2\nabla\hat{\phi} = 0.$$

Since $\hat{\Omega}\backslash W \subset \hat{U} \subset \hat{\Omega}$, we get

$$\int_C qw^2 = R\int_{\hat{U}}(\nabla\hat{\phi})\hat{w}^2 = R\sum_i \int_{\hat{U}_i}(\nabla\hat{\phi})\hat{w}^2. \tag{3.15}$$

We now use Lemma 3.1 to get η_i continuous such that

$$\int_{\hat{U}_i}(\nabla\hat{\phi})\hat{w}^2 = \int_{\hat{U}_i}(\nabla\hat{\phi})\eta_i^2(\hat{\phi}).$$

We define the function F_i by $F_i' = \eta_i^2$ and $F_i(0) = 0$, and we obtain

$$\int_{\hat{U}_i}(\nabla\hat{\phi})\hat{w}^2 = \int_{\hat{U}_i}\nabla F_i(\hat{\phi}).$$

If we define $\hat{U}_i^\varepsilon := \{\hat{x} \in \hat{U}_i \text{ such that } \inf_{\hat{U}_i}\hat{\phi} + \varepsilon < \hat{\phi}(x) < \sup_{\hat{U}_i}\hat{\phi} - \varepsilon\}$, then it follows from the dominated convergence theorem that

$$\int_{\hat{U}_i^\varepsilon}(\nabla\hat{\phi})\hat{w}^2 \underset{\varepsilon\to 0}{\longrightarrow} \int_{\hat{U}_i}(\nabla\hat{\phi})\hat{w}^2. \tag{3.16}$$

We now prove (i) $\Rightarrow$ (ii) by contraposition. We suppose that there exist no periodic unbounded trajectories of q. In $\hat{U}_i$, the trajectories of q are exactly the level sets of $\hat{\phi}$. We consider the following set: $U_i^\varepsilon := \Pi^{-1}(\hat{U}_i^\varepsilon)$. Let $x_0 \in U_i^\varepsilon$ and let $U_{i,0}^\varepsilon$ be the connected component of U_i^ε containing x_0. We claim that Π is a surjection from $U_{i,0}^\varepsilon$ to $\hat{U}_i^\varepsilon$. For that purpose we prove that $\Pi(U_{i,0}^\varepsilon)$ is open and closed in $\hat{U}_i^\varepsilon$.

Let $\hat{x} \in \Pi(U_{i,0}^\varepsilon)$, and $r > 0$ be sufficiently small to have $B(\hat{x}, r) \subset \hat{U}_i^\varepsilon$. Let $x \in U_{i,0}^\varepsilon$ such that $\Pi(x) = \hat{x}$; then $\Pi(B(x, r)) = B(\hat{x}, r)$, which leads to $B(\hat{x}, r) \subset \Pi(U_{i,0}^\varepsilon)$, so $\Pi(U_{i,0}^\varepsilon)$ is open in $\hat{U}_i^\varepsilon$.

On the other hand, let $\{\hat{x}_n\}$ be a sequence of $\Pi(U_{i,0}^\varepsilon)$ converging to $\hat{x} \in \hat{U}_i^\varepsilon$. Since $\hat{U}_i^\varepsilon$ is open, for $r > 0$ sufficiently small $B(\hat{x}, r) \subset \hat{U}_i^\varepsilon$. Hence, for n large enough, we have $\hat{x}_n \in B(\hat{x}, r)$, so there exists $N \in \mathbb{N}$ and $s > 0$ such that $B(\hat{x}_N, s) \subset B(\hat{x}, r)$ and $\hat{x} \in B(\hat{x}_N, s)$. Finally, let $x_N \in U_{i,0}^\varepsilon$ such that $\Pi(x_N) = \hat{x}_N$; then there exists $x \in B(x_N, s)$ such that $\Pi(x) = \hat{x}$, which leads to the conclusion that $\Pi(U_{i,0}^\varepsilon)$ is closed in $\hat{U}_i^\varepsilon$.

Now, by definition, $\hat{U}_i$ and $\hat{U}_i^\varepsilon$ only contain "regular" trajectories of q, and so does $U_{i,0}^\varepsilon$. By assumption, there exist no periodic unbounded trajectories of q, so all the trajectories of q in $U_{i,0}^\varepsilon$ are bounded. Moreover, $\partial U_{i,0}^\varepsilon$ is the disjoint reunion of two bounded, regular trajectories of q. All the trajectories of q in $U_{i,0}^\varepsilon$ are level sets of ϕ, and by a compactness argument we get that $U_{i,0}^\varepsilon$ is bounded in Ω.

From this boundedness, we obtain that $\Pi : U_{i,0}^{\varepsilon} \to \hat{U}_i^{\varepsilon}$ is a bijection. Indeed, if it is not injective, since $U_{i,0}^{\varepsilon}$ is connected we would have a path connecting two different points x_1 and x_2 of $U_{i,0}^{\varepsilon}$, such that $\Pi(x_1) = \Pi(x_2)$, and by periodicity, $U_{i,0}^{\varepsilon}$ could not be bounded.

We conclude that $\Pi : U_{i,0}^{\varepsilon} \to \hat{U}_i^{\varepsilon}$ is a measure-preserving bijection, by definition of the measure on T. We get then

$$\int_{\hat{U}_i^{\varepsilon}} (\nabla\hat{\phi})\hat{w}^2 = \int_{U_{i,0}^{\varepsilon}} (\nabla\phi)w^2 = \int_{U_{i,0}^{\varepsilon}} \nabla F_i(\phi) = \int_{\partial U_{i,0}^{\varepsilon}} F_i(\phi)\mathbf{n},$$

where $\mathbf{n}$ is the unit outward normal vector field to $\partial U_{i,0}^{\varepsilon}$. Finally $\partial U_{i,0}^{\varepsilon}$ is the union of two level sets C_1 and C_2 of ϕ in Ω, which are both simple closed curves, so we can write

$$\int_{U_{i,0}^{\varepsilon}} (\nabla\phi)w^2 = F_i(\phi(C_1)) \int_{C_1} \mathbf{n} + F_i(\phi(C_2)) \int_{C_2} \mathbf{n},$$

with

$$\int_{C_1} \mathbf{n} = \int_{C_2} \mathbf{n} = 0,$$

because the integral of the unit normal on a C^1 closed curve in $\mathbb{R}^2$ is zero. Therefore, using (3.15) and (3.16), we get $\int_C qw^2 = 0$, for all $w \in \mathcal{I}$.

We now prove (ii) $\Rightarrow$ (i). Let $x \in \Omega$ and $\mathbf{a} \neq 0$ such that $T(x)$ is $\mathbf{a}$-periodic and unbounded such that $|\mathbf{a}|$ is minimal. Let $\hat{U}_i$ be the connected component of $\hat{U}$ containing $\hat{x} := \Pi(x)$. We define as previously $\hat{U}_i^{\varepsilon}$ for ε sufficiently small in order to have $\hat{U}_i^{\varepsilon} \neq \emptyset$. Let $x_0 \in \Pi^{-1}(\hat{U}_i^{\varepsilon})$; we define $U_{i,0}^{\varepsilon}$ to be the connected component of $\Pi^{-1}(\hat{U}_i^{\varepsilon})$ containing x_0. Let $e_1' := \mathbf{a}/|\mathbf{a}|$, and e_2' be such that (e_1', e_2') is an orthonormal frame of $\mathbb{R}^2$. We set

$$U_i^{\varepsilon} := \{x = x_1'e_1' + x_2'e_2' \in U_{i,0}^{\varepsilon} \text{ such that } 0 \le x_1' < |\mathbf{a}|\}.$$

Using similar arguments to (i) $\Rightarrow$ (ii), we get that Π is a measure-preserving bijection from U_i^{ε} to $\hat{U}_i^{\varepsilon}$, and we have

$$\int_{\hat{U}_i^{\varepsilon}} (\nabla\hat{\phi})\hat{w}^2 = \int_{U_i^{\varepsilon}} (\nabla\phi)w^2 = \int_{U_i^{\varepsilon}} \nabla F_i(\phi) = \int_{\partial U_i^{\varepsilon}} F_i(\phi)\mathbf{n}.$$

The boundary of U_i^{ε} consists then of the connected pieces C_1 and C_2 of level sets of ϕ, and two segments, which are $S_1 := U_{i,0}^{\varepsilon} \cap \{x_1' = 0\}$ and $S_2 := U_{i,0}^{\varepsilon} \cap \{x_1' = |\mathbf{a}|\}$. We have $S_1 = S_2 + \mathbf{a}$. By periodicity of ϕ, and the fact that the outward unit normal vector on S_1 is the opposite of the outward unit normal vector on S_2, we get

$$\int_{S_1} F_i(\phi)\mathbf{n} + \int_{S_2} F_i(\phi)\mathbf{n} = 0.$$

Moreover, we have

$$\int_{C_1} F_i(\phi)\mathbf{n} = F_i(\phi(C_1)) \int_{C_1} \mathbf{n} = F_i(\phi(C_1))R\mathbf{a},$$

where R is still the matrix of a rotation of angle $\pi/2$. Besides,

$$\int_{C_2} F_i(\phi)\mathbf{n} = F_i(\phi(C_2)) \int_{C_2} \mathbf{n} = -F_i(\phi(C_2))R\mathbf{a}.$$

It suffices now to consider a function F_i defined by $F_i' = \eta_i^2$ which is not constant on U_i^ε, so that we just need to consider a function η_i which has compact support in $\phi(U_i^\varepsilon)$, but is not identically zero. This way we get $F_i(\phi(C_2)) \neq F_i(\phi(C_1))$, and we obtain

$$\int_{U_i^\varepsilon} \nabla F_i(\phi) \neq 0.$$

We set then $w_0 = \eta_i(\phi)$ on U_i^ε, and 0 otherwise. The function w_0 obviously belongs to $\mathcal{I}$, and using (3.15), all the terms in the sum vanish except for the integral on $\hat{U}_i$, so $\int_C q w_0^2 \neq 0$. This proves (ii).

In the last part of the theorem, we need to prove that whenever $\int_C q w_0^2 \neq 0$, it is proportional to $\mathbf{a}$, where $\mathbf{a}$ is such that all the unbounded periodic trajectories of q in Ω are $\mathbf{a}$-periodic.

For that purpose, we return to the previous computations. We know that for $\varepsilon > 0$ sufficiently small we have

$$\int_{\hat{U}_i^\varepsilon} (\nabla\hat{\phi})\hat{w}^2 = (F_i(\phi(C_1)) - F_i(\phi(C_2)))R\mathbf{a}.$$

Hence, for any $\varepsilon > 0$ we have

$$\mathbf{a} \cdot \int_{\hat{U}_i^\varepsilon} (\nabla\hat{\phi})\hat{w}^2 = 0,$$

which remains true at the limit $\varepsilon \to 0$. Using (3.15) we get

$$\mathbf{a} \cdot \int_C \nabla\phi w^2 = 0,$$

which gives

$$R\mathbf{a} \cdot \int_C qw^2 = 0,$$

for any $w \in \mathcal{I}$. This is equivalent to saying $\int_C qw^2 \in \mathbb{R}\mathbf{a}$. □

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