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The principal eigenvalue of a mixed local and nonlocal operator with drift [☆]

Craig Cowan [a], Mohammad El Smaily [b,*], Pierre Aime Feulefack [b]

[a] Department of Mathematics, University of Manitoba, Winnipeg, MB, Canada

[b] Department of Mathematics and Statistics, University of Northern British Columbia, Prince George, BC, Canada

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Abstract

We study the eigenvalue problem involving the mixed local-nonlocal operator $L := -\Delta + (-\Delta)^s + q \cdot \nabla + a(x)\mathrm{Id}$ in a bounded domain $\Omega \subset \mathbb{R}^N$, where a Dirichlet condition is posed on $\mathbb{R}^N \setminus \Omega$. The vector field q stands for a drift or advection in the medium. We prove the existence of a principal eigenvalue and a principal eigenfunction for $s \in (0, 1/2]$. Moreover, we prove $C^{2,\alpha}$ regularity, up to the boundary, of the solution to the problem $Lu = f$ when coupled with a Dirichlet condition and $0 < s < 1/2$. To prove the regularity and the existence of a principal eigenvalue, we use the L^p theory for L obtained via a continuation argument, Krein-Rutman theorem as well as a Hopf Lemma and a maximum principle for the operator L, which we derive in this paper.

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* Corresponding author.

E-mail addresses: craig.cowan@umanitoba.ca (C. Cowan), mohammad.elsmaily@unbc.ca (M. El Smaily), pierre.feulefack@aims-cameroon.org (P.A. Feulefack).

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Contents

1. Introduction and main results

The study of the principal eigenvalue of an operator is essential for many important results in the analysis of elliptic and parabolic PDE as well as the analysis of elliptic and parabolic integro-differential equations (IDE). For instance, the principal eigenvalue is fundamental in the study of semi-linear problems [8,12], bifurcation theory, stability analysis of equilibrium of reaction-diffusion [5,6], large deviation principle, and risk-sensitive control [2]. The principal eigenvalue of an operator also plays a role in determining whether the maximum principle holds or not for the operator at hand [7,17,24].

We are interested in the study of the principal eigenvalue for an operator involving an advection term (or drift) and a mixed local (elliptic) and nonlocal operator. We consider the following problem

$$\begin{cases} Lu := -\Delta u + (-\Delta)^s u + q \cdot \nabla u + a(x)u = \lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where, for some $\alpha \in (0, 1)$,

$$\Omega \text{ is an open bounded domain of } \mathbb{R}^N \text{ with } \mathcal{C}^{2,\alpha} \text{ boundary.} \tag{1.2}$$

The operator L is an elliptic operator (in general non-self-adjoint, unless $q \equiv 0$) obtained by the superposition of the classical and the fractional Laplacian $(-\Delta)^s$ where $s \in (0, 1/2]$. Problem (1.1) has also an advection term $q \cdot \nabla u$, where

$$q : \Omega \to \mathbb{R}^N \text{ is a vector field in the Hölder space } \mathcal{C}^{0,\alpha}(\overline{\Omega}). \tag{1.3}$$

The vector field q can be viewed as a transport flow in (1.1). The function a in (1.1) is assumed to satisfy

$$a \in \mathcal{C}^{0,\alpha}(\overline{\Omega}). \tag{1.4}$$

We recall that the operator $(-\Delta)^s$, $s \in (0,1)$, stands for the fractional Laplacian and it is defined, for a compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$ of class $\mathcal{C}^2$, by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy.$$

The constant $C_{N,s}$ in the above definition is given by

$$C_{N,s} := \pi^{-\frac{N}{2}} 2^{2s} s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(1-s)},$$

and it is chosen so that the operator $(-\Delta)^s$ is equivalently defined by its Fourier transform

$$\mathcal{F}((-\Delta)^s u) = |\cdot|^{2s} \mathcal{F}(u).$$

It is known that we have the following limits

$$\lim_{s \to 0^+} (-\Delta)^s u = u \quad \text{and} \quad \lim_{s \to 1^-} (-\Delta)^s u = -\Delta u \quad \text{for } u \in \mathcal{C}_c^2(\mathbb{R}^N).$$

Definition 1.1. By the principal eigenvalue of L, we mean a value $\lambda_1 \in \mathbb{R}$ for which (1.1) admits a positive solution u ($u > 0$) in Ω. Throughout the paper, we will denote by $\lambda_1 := \lambda_1(\Omega, q)$ the first eigenvalue of L in Ω with a Dirichlet condition on $\mathbb{R}^N \setminus \Omega$. We will denote by φ_1 the corresponding unique (up to multiplication by a nonzero real) eigenfunction of L. We will refer to φ_1, which will have a constant sign over Ω, as the principal eigenfunction of L.

The interest in the study of problems involving mixed local-nonlocal operator has been growing rapidly in recent years. This is due to their ability to describe the superstition of two stochastic processes with different scales (Brownian motion and Lévy process) [16]. The mixed local-nonlocal operator in the form (without advection and without a zero order term)

$$L_0 := -\Delta + (-\Delta)^s, \quad s \in (0,1),$$

has received by far great attention from different points of view. This includes existence and non-existence results [1,4,8,10,21,23], regularity results [13,18,22,28,31], associated eigenvalue problems [9,12,14,26,29], and radial symmetry results [11].

In this paper, we consider a mixed local-nonlocal operator with the *additional advection term* $q \cdot \nabla$, where $q \in L^\infty(\Omega)$ is a bounded vector field. We aim to study the existence of the principal eigenvalue and the corresponding eigenfunction in Ω for $L := -\Delta + (-\Delta)^s + q \cdot \nabla + a(x)Id$ with $s \in (0, 1/2]$. To the best of our knowledge, the presence of an advection term has not been addressed before.

It is important to note that when the operator L does not include an advection term, that is $L \equiv L_0$, the operator is self-adjoint and the study of the principal eigenvalue for L_0 relies on a variational characterization via the Rayleigh quotient (see [8,9,14]). Namely,

$$\lambda_1(\Omega) := \inf_{u \in \mathcal{X}_0^s(\Omega) \setminus \{0\}} \frac{[u]_{\mathcal{X}^s(\Omega)}}{\|u\|_{L^2(\Omega)}^2}, \tag{1.5}$$

where the space $\mathcal{X}_0^s(\Omega)$ and the semi-norm $[\cdot]_{\mathcal{X}^s(\Omega)}$ are defined in Section 2, below. However, in the presence of advection, the operator L is *no longer self-adjoint* and so there is no simple variational formulation for the first eigenvalue as in (1.5). We will prove the existence of such principal eigenvalue of L and the corresponding eigenfunction with the aid of the Krein-Rutman theorem (see [15]). We will use the version of Krein-Rutman theorem stated in [17, Theorem 1.2] and we recall it in Theorem A below.

Lastly, we mention that integro-differential equations arise naturally in the study of stochastic processes with jumps. They describe a biological species whose individuals diffuse either by a random walk or by a jump process according to the prescribe probabilities [31]. The generator of a Lévy process has the following general structure [27]

$$\mathcal{L}u := \sum_{i,j=1} a_{ij} D_{ij} u + \sum_{j=1} q_j D_j u + a(x)u + PV. \int_{\mathbb{R}^N} (u(x) - u(y))K(x-y)\, dy, \tag{1.6}$$

where K is a measurable kernel on $\mathbb{R}^N$ satisfying $\int_{\mathbb{R}^N} \min\{1, |z|^2\} K(z)\, dz < \infty$ and $a(x)u$ is a zero order term. The first and second terms in (1.6) correspond to the diffusion and the drift respectively [14]. The study of the operator $\mathcal{L}$ in (1.6) with all its components (diffusion, drift, zero order term and jump) is quite intriguing. By far, there are just a few contributions in this direction. In [2], while studying the risk-sensitive control for a class of diffusion with jumps, the authors investigated the existence of the principal eigenvalue for the class of operators $\mathcal{L}$ where the kernel is locally integrable. In [3], the authors also considered a locally integrable kernel and proved the existence of generalized principal eigenvalue in $\mathbb{R}^N$. We refer to [24, Chap. 3] where elliptic problems involving general second order elliptic integro-differential operator have been considered. Note that our operator L in (1.1) corresponds to $a_{ij} = \delta_{ij}$ in (1.6). In this present work, we only consider an L where the nonlocal operator is replaced by the fractional Laplacian.

We state our first result as follows.

Theorem 1.2. *Assume that q and a satisfy (1.3) and (1.4) respectively and let $s \in (0, 1/2]$. Then, there exists a principal eigenpair $(\lambda_1(\Omega, q), \varphi_1)$ for the problem (1.1) such that*

- **(a)** *$\lambda_1(\Omega, q)$ is an eigenvalue of L in Ω and the corresponding eigenfunction φ_1 has a constant sign in Ω and it is unique up to multiplication by a nonzero constant. Moreover, φ_1 satisfies $\partial_\nu \varphi_1 < 0$ on $\partial\Omega$, where ν stands for the outward normal on $\partial\Omega$.*
- **(b)** *For all $s \in (0, 1/2]$, we have $\varphi_1 \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \cap C(\mathbb{R}^N)$. Moreover, if $s \in (0, 1/2)$ and $0 < \alpha < 1 - 2s$, the principal eigenfunction satisfies $\varphi_1 \in C^{2,\alpha}(\overline{\Omega})$.*
- **(c)** *If $\lambda \in \mathbb{R}$ is a real eigenvalue of L and $\lambda \neq \lambda_1(\Omega, q)$, then $\lambda > \lambda_1(\Omega, q)$.*
- **(d)** *If $\lambda \in \mathbb{C}$ is an eigenvalue of L, and $\lambda \neq \lambda_1(\Omega, q)$, then $\Re(\lambda) > \lambda_1(\Omega, q)$, where $\Re(\lambda)$ is the real part of λ.*
- **(e)** *The principal eigenvalue $\lambda_1(\Omega, q)$ is characterized by the following min-max formula*

$$\lambda_1(\Omega, q) = \max_{u \in \mathcal{V}(\Omega)} \inf_{x \in \Omega} \frac{Lu}{u}, \tag{1.7}$$

where

$$\mathcal{V}(\Omega) := \Big\{ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap C_c(\mathbb{R}^N) : \ u > 0 \text{ in } \Omega \ \text{ and } \ u \equiv 0 \text{ on } \ \mathbb{R}^N \setminus \Omega \Big\}.$$

We also prove the following Hopf Lemma for the operator L. We emphasize that the result holds for any $s \in (0,1)$. We will use the following result in proving Theorem 1.2 but we will state the result in a general setting.

Theorem 1.3 *(Hopf Lemma). Assume that q and a satisfy (1.3) and (1.4) respectively. Assume furthermore that $a(x) \geq 0$ for all $x \in \Omega$. Let $s \in (0,1)$. Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain and let $c_0 \in \mathbb{R}$ such that*

$$c_0 a(x) \leq 0, \; x \in \Omega. \tag{1.8}$$

Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\mathbb{R}^N) \cap \mathcal{C}^1(\overline{\Omega})$ such that u is bounded in $\mathbb{R}^N$ and

$$Lu := -\Delta u + (-\Delta)^s u + q(x) \cdot \nabla u + a(x)u \geq 0 \text{ in } \quad \Omega. \tag{1.9}$$

Let $x_0 \in \partial\Omega$. Assume that $u(x) = c_0$ on $B_{R_0}(x_0) \cap \partial\Omega$, for some $R_0 > 0$, and that $u \geq c_0$ in $\mathbb{R}^N$. If $u \not\equiv c_0$ in $\mathbb{R}^N$, then

$$\partial_\nu u(x_0) < 0, \tag{1.10}$$

where ν denotes the outer unit normal to $\partial\Omega$ at x_0.

Remark 1.4. Observe that the condition (1.8) is not needed whenever $a \equiv 0$ in Ω. In other words, c_0 can be an arbitrary constant in $\mathbb{R}$ when $a \equiv 0$.

We also have the following remark:

Remark 1.5. Note that Theorem 1.3 holds for all $0 < s < 1$ and that the function u is not assumed to be $\mathcal{C}^2(\overline{\Omega})$. We only assume that u is differentiable at $x_0 \in \partial\Omega$. We will state another version of the Hopf Lemma in Theorem 3.2 below. However, the other version requires more regularity on u. The proof of Theorem 3.2 turns out to be shorter because it relies on an inequality satisfied by $(-\Delta)^s u$ in a neighborhood of x_0. As $\mathcal{C}^2$ regularity, up to the boundary, is not confirmed for $s = 1/2$, we will see that Theorem 1.3 turns out to be more helpful, than Theorem 3.2, in proving Theorem 1.2.

The following three theorems address the regularity of solutions to the linear problem

$$\begin{cases} Lu := -\Delta u + (-\Delta)^s u + q \cdot \nabla u + au = f & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.11}$$

We mention that the regularity of u—up to the boundary of Ω, occurs for $0 < s < 1/2$ and $0 < \alpha < 1 - 2s$. In general, for $s \in (0, 1/2]$, we prove an interior regularity result.

Theorem 1.6 *($\mathcal{C}^{2,\alpha}$ interior regularity). Let a, f and q be in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. Assume that $s \in (0, 1/2]$ and that $u \in W^{2,p}(\Omega)$ is a solution of (1.11). Then $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega'})$ for every $\Omega' \subset\subset \Omega$. Moreover, there exists a constant $C > 0$ such that*

$$\|u\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega'})} \leq C \|f\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} \tag{1.12}$$

The following theorem provides a $W^{2,p}$ estimate for the solution to the mixed local/nonlocal problem (1.1).

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $s \in (0, 1/2]$. Assume that $f \in L^p(\Omega)$ with $1 < p < \infty$, $a \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ satisfies $a(x) \geq 0$ in Ω and $q \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. Then, the problem (1.11) has a unique solution $u \in W^{2,p}(\Omega)$. Furthermore, there exists a constant $C := C(N, s, p, \Omega) > 0$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{1.13}$$

As a consequence of Theorem 1.7, we have the following $\mathcal{C}^{2,\alpha}$ regularity, up to the boundary, for (1.11) when $0 < s < 1/2$.

Theorem 1.8 *(Hölder regularity up to the boundary when $s < 1/2$). Let $s \in (0, \frac{1}{2})$ and $0 < \alpha < 1 - 2s$. Assume that the advection term satisfies $q \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and that $a \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ satisfies $a(x) \geq 0$ in Ω. Then, there is some $C > 0$ such that for all $f \in C^{0,\alpha}(\overline{\Omega})$ there is some $u \in C^{2,\alpha}(\overline{\Omega})$ that satisfies*

$$\begin{cases} Lu = -\Delta u + (-\Delta)^s u + q \cdot \nabla u + au = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.14}$$

Moreover, we have

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{0,\alpha}(\overline{\Omega})}.$$

We briefly comment on the proof of Theorem 1.2. The main ingredient in the proof of Theorem 1.2 is the Krein-Rutman theorem, which relies on the strong maximum principle for the operator L and the L^p-theory of the problem (1.11) (see Theorem 1.7).

Indeed, with the aid of the L^p-theory of L_0 developed in [31, Theorem 1.4] and for more general second order elliptic integro-differential operators developed in [24, Theorem 3.1.23], we first prove using the method of continuity (see [25, Theorem 5.2]) that for $f \in L^p(\Omega)$, there exists a unique solution $u \in W^{2,p}(\Omega)$ of problem (1.11) for any $1 < p < \infty$. Then, using the Sobolev (Morrey) embedding theorem we in fact have that $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$, for any $\beta \in (0, 1)$. Exploiting the fact that $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$, we obtain an interior $\mathcal{C}^{2,\alpha}$ regularity for u in Theorem 1.6 for all $s \in (0, 1/2]$ thanks to the regularity result of [30, Proposition 2.7]. This allows us to apply the strong maximum principle for L in Theorem 3.1 and thus complete the proof of Theorem 1.7.

We point out that since our strategy of proving Theorem 1.2 relies on the L^p-theory of problem (1.11), combined with the application of the Krein-Rutmen theorem, our result holds for more general mixed local-nonlocal operators satisfying the strong maximum principle as given in (1.6) with the kernel of K satisfying

$$\frac{\kappa_1}{|x-y|^{N+2s}} \leq K(x-y) \leq \frac{\kappa_2}{|x-y|^{N+2s}}, \qquad \kappa_1, \kappa_2 > 0, s \in (0, 1/2]. \tag{1.15}$$

We refer the interested reader to [24, Chap. 3] for general second order elliptic integro-differential operators satisfying such properties. In particular, any nonlocal operator of small order will satisfy (1.15) (see [19,20] and the references therein).

The rest of this paper is organized as follows. In Section 2, we introduce some functional spaces. In Section 3, we prove the strong maximum principle and the Hopf Lemma for L. In Section 4, we develop the L^p-theory for L and prove the existence and uniqueness of solution $u \in W^{2,p}(\Omega)$ to problem (1.11). Section 5 is devoted to the proof of Theorem 1.2 by using most of the results in the previous sections.

2. Functional setting

We start this section by fixing some notation. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. For the vector field $q : \Omega \to \mathbb{R}^N$, we write $q \in L^\infty(\Omega)$ (resp. $q \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$) whenever $q_j \in L^\infty(\Omega)$ (resp. $q_j \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$), $j = 1, 2, \cdots, N$. We denote by $\mathcal{C}^{k,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, the Banach space of functions $u \in \mathcal{C}^k(\overline{\Omega})$ such that derivative of order k belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ with the norm

$$\|u\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} := \|u\|_{\mathcal{C}^k(\overline{\Omega})} + \sum_{|\tau|=k} [D^\tau u]_{\mathcal{C}^{0,\alpha}(\overline{\Omega})},$$

where

$$[u]_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} = \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

and $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ is the Banach space of functions $u \in \mathcal{C}^0(\overline{\Omega})$ which are Hölder continuous with exponent α and the norm $\|u\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}$.

If $k \in \mathbb{N}$, as usual we set

$$W^{k,p}(\Omega) := \Big\{ u \in L^p(\Omega) \ : \ \ D^\alpha u \text{ exists for all } \alpha \in \mathbb{N}^N,\ |\alpha| \leq k \text{ and } u \in L^p(\Omega) \ \Big\}$$

for the Banach space of k-times (weakly) differentiable functions in $L^p(\Omega)$. Moreover, in the fractional setting, for $s \in (0, 1)$ and $p \in [1, \infty)$, we set

$$W^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) \ : \ \frac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \Big\}.$$

The space $W^{s,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\|u\|^p_{L^p(\Omega)} + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dxdy \Big)^{\frac{1}{p}}.$$

We also define the space $\mathcal{X}^s(\Omega)$ by

$$\mathcal{X}^s(\Omega) := \Big\{ u \in L^2(\Omega) : \quad u|_\Omega \in H^1(\Omega); \quad [u]_{\mathcal{X}^s(\Omega)} < \infty \Big\},$$

where the corresponding Gagliardo seminorm $[\cdot]_{\mathcal{X}^s(\Omega)}$ is given by

$$[u]_{\mathcal{X}^s(\Omega)} := \int\limits_{\Omega} |\nabla u|^2 \, dx + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy.$$

Note that the space $\mathcal{X}^s(\Omega)$ is a Hilbert space when furnished with the scalar product

$$\langle u, v\rangle_{\mathcal{X}^s(\Omega)} := \int\limits_{\Omega} uv \, dx + \int\limits_{\Omega} \nabla u \cdot \nabla v \, dx + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy$$

and the corresponding norm is given by $\|u\|_{\mathcal{X}^s(\Omega)} = \sqrt{\langle u, v\rangle_{\mathcal{X}^s(\Omega)}}$. Define

$$\mathcal{X}^s_0(\Omega) := \big\{u \in \mathcal{X}^s(\Omega) : \quad u \equiv 0 \quad \text{on} \quad \mathbb{R}^N \setminus \Omega\big\}.$$

Note that if $u \in \mathcal{X}^s_0(\Omega)$ then $u|_\Omega \in H^1_0(\Omega)$ due to the regularity assumption of $\partial\Omega$.

Finally, we define the space $\mathcal{L}^1_s(\mathbb{R}^N)$ by

$$\mathcal{L}^1_s(\mathbb{R}^N) := \big\{u : \mathbb{R}^N \to \mathbb{R}, \text{ such that } u \text{ is measurable and } \|u\|_{\mathcal{L}^1_s(\mathbb{R}^N)} < \infty\big\},$$

where

$$\|u\|_{\mathcal{L}^s_1(\mathbb{R}^N)} := \int\limits_{\mathbb{R}^N} \frac{|u(y)|}{1+|y|^{N+2s}} dx.$$

3. Hopf lemma, maximum principle and interior regularity: proof of Theorem 1.3 and Theorem 1.6

In this section, we derive some results for the operator L. These will be important in the proof of Theorem 1.2. We start with the following result on the strong maximum principle for L.

Theorem 3.1 *(Strong maximum principle). Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, q and a be in $L^\infty(\Omega)$ with $a(x) \geq 0$ in Ω. Let $s \in (0,1)$ and $u \in \mathcal{L}^1_s(\mathbb{R}^N)$ be a function in $C^2(\Omega) \cap C(\mathbb{R}^N)$ that satisfies*

$$\begin{cases} Lu & \geq 0 \;\; in \quad \Omega \\ u & \geq 0 \;\; on \quad \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then $u > 0$ in Ω or $u \equiv 0$ in $\mathbb{R}^N$.

Proof of Theorem 3.1. Suppose to the contrary that u is not positive in Ω. Since Ω is bounded, $\overline{\Omega}$ is compact. Since u is continuous in $\mathbb{R}^N$ and $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$, there is a point $x_0 \in \Omega$ with

$$u(x_0) = \min_{x \in \overline{\Omega}} u(x) \leq 0. \tag{3.1}$$

Therefore, as x_0 is an interior point where the minimum of u is attained, it follows that $q \cdot \nabla u(x_0) = 0$ and $\Delta u(x_0) \geq 0$. Hence, from the definition of the operator L, and since $a(x) \geq 0$ in Ω, we have that

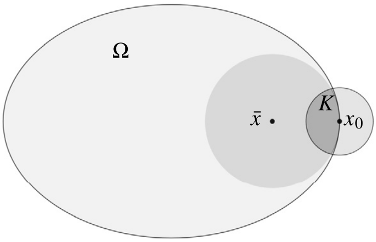

Fig. 1. The open set $K \subset \Omega$ is the intersection of the ball centered at x_0 with the ball centered at $\bar{x}$, which is tangent to $\partial\Omega$ at x_0. Note that $\overline{K} \cap \partial\Omega = \{x_0\}$.

$$(-\Delta)^s u(x_0) \geq \Delta u(x_0) - a(x_0)u(x_0) \geq 0.$$

Whereas by (3.1), we have that $u(x_0) \leq u(x)$ for all $x \in \mathbb{R}^N$. It follows that

$$0 \leq (-\Delta)^s u(x_0) = P.V. \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}}\, dy = \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}}\, dy \leq 0.$$

Moreover, since the integrand is non-positive by assumption and (3.1), we conclude that

$$u \equiv u(x_0) \quad \text{in} \quad \mathbb{R}^N.$$

Now, since $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$, it follows that $u \equiv 0$ in Ω and therefore $u \equiv 0$ in $\mathbb{R}^N$. This leads to a contradiction and the proof is established. □

We now prove the Hopf Lemma stated in Theorem 1.3, for all $s \in (0, 1)$.

Proof of Theorem 1.3. Let be $B_r(\bar{x})$ a ball centered at $\bar{x} \in \Omega$ that touches $\partial\Omega$ at $x_0 \in \partial\Omega$. Let K be the set defined by

$$K := B_r(\bar{x}) \cap B_{\frac{r}{2}}(x_0).$$

We introduce the auxiliary function

$$v(x) := e^{-\theta \mathcal{K}(x)} - e^{-\theta(1+r^2)^{\frac{1}{2}}}, \qquad \text{where} \qquad \mathcal{K}(x) := (|x - \bar{x}|^2 + 1)^{\frac{1}{2}}$$

and θ is a positive constant to be chosen later. (Fig. 1.)

We have

$$v > 0 \quad \text{in} \quad B_r(\bar{x}), \qquad v = 0 \;\text{ on }\; \partial B_r(\bar{x}) \quad \text{ and } \quad v < 0 \;\text{ on }\; \mathbb{R}^N \setminus B_r(\bar{x}).$$

For any $x \in \mathbb{R}^N$, computation shows that

$$\begin{aligned} Lv(x) &= -\Delta v(x) + (-\Delta)^s v(x) + q(x)\cdot\nabla v(x) + a(x)v(x) \\ &= e^{-\theta\mathcal{K}(x)}\left(\frac{\theta N}{\mathcal{K}(x)} - \theta\frac{q(x)\cdot(x-\bar{x})}{\mathcal{K}(x)} - |x-\bar{x}|^2\Big(\frac{\theta}{\mathcal{K}^3(x)} + \frac{\theta^2}{\mathcal{K}^2(x)}\Big)\right) \\ &\quad + (-\Delta)^s v(x) + a(x)v(x). \end{aligned} \tag{3.2}$$

Observe that $1 - e^{-\rho} \le \rho$ for all $\rho \in \mathbb{R}$. Therefore,

$$\begin{aligned} (-\Delta)^s v(x) &= \frac{C_{N,s}}{2}\int_{\mathbb{R}^N} \frac{2v(x) - v(x+y) - v(x-y)}{|y|^{N+2s}}\,dy \\ &= \frac{C_{N,s}e^{-\theta\mathcal{K}(x)}}{2}\int_{\mathbb{R}^N} \frac{2 - e^{-\theta(\mathcal{K}(x+y)-\mathcal{K}(x))} - e^{-\theta(\mathcal{K}(x-y)-\mathcal{K}(x))}}{|y|^{N+2s}}\,dy \\ &\le \frac{C_{N,s}\theta e^{-\theta\mathcal{K}(x)}}{2}\int_{\mathbb{R}^N} \frac{(\mathcal{K}(x+y)-\mathcal{K}(x)) + (\mathcal{K}(x-y)-\mathcal{K}(x))}{|y|^{N+2s}}\,dy \\ &= -\theta e^{-\theta\mathcal{K}(x)}(-\Delta)^s\mathcal{K}(x). \end{aligned} \tag{3.3}$$

We see that

$$1 \le \mathcal{K}(x) \le (\mathrm{diam}(\Omega)^2 + 1)^{1/2} := \Lambda. \tag{3.4}$$

The constant Λ in (3.4) depends only on the domain Ω. We now compute

$$\begin{aligned} &\frac{2|(-\Delta)^s\mathcal{K}(x)|}{C_{N,s}} \\ &\le \int_{B_1} \frac{|(\mathcal{K}(x+y)-\mathcal{K}(x)) + (\mathcal{K}(x-y)-\mathcal{K}(x))|}{|y|^{N+2s}}\,dy + \int_{\mathbb{R}^N\setminus B_1} \frac{2\mathcal{K}(x)}{|y|^{N+2s}}\,dy \\ &\le \int_{B_1}\int_0^1 \frac{|\langle\nabla\mathcal{K}(x+ty)) - \nabla\mathcal{K}(x-ty), y\rangle|}{|y|^{N+2s}}\,dtdy + \int_{\mathbb{R}^N\setminus B_1} \frac{2\Lambda}{|y|^{N+2s}}\,dy \\ &\le \int_{B_1}\int_0^1 \frac{|\nabla\mathcal{K}(x+ty)) - \nabla\mathcal{K}(x-ty)|}{|y|^{N+2s-1}}\,dtdy + \frac{2\omega_{N-1}\Lambda}{2s}. \end{aligned}$$

We also have

$$\begin{aligned} |\nabla\mathcal{K}(x+ty)) - \nabla\mathcal{K}(x-ty)| &= \left|\frac{(x+ty)}{\mathcal{K}(x+ty))} - \frac{(x-ty)}{\mathcal{K}(x-ty)}\right| \\ &= \left|ty\left(\frac{1}{\mathcal{K}(x+ty)} + \frac{1}{\mathcal{K}(x-ty)}\right) + x\left(\frac{1}{\mathcal{K}(x+ty)} - \frac{1}{\mathcal{K}(x-ty)}\right)\right| \end{aligned}$$

$$\le 2t|y| + \left|\frac{1}{\mathcal{K}(x+ty)} - \frac{1}{\mathcal{K}(x-ty)}\right||x|, \text{ as } \mathcal{K} \ge 1$$

$$\le 2t|y|\left(1 + \left|\nabla\left(\frac{1}{\mathcal{K}(z)}\right)\right||x|\right), \text{ for some } z \in [x-ty, x+ty]$$

$$\le 2t|y|\left(1 + \left|\frac{z-\bar{x}}{2\mathcal{K}^3(z)}\right||x|\right) \le 2t|y|\,(1 + |z-\bar{x}|\,|x|)$$

$$\le 2t|y|\,(1 + |z-x|\,|x| + |\bar{x}|\,|x|) \le 2t|y|\,(1 + (3t|y| + |\bar{x}|)|x|).$$

Since $x, \bar{x} \in \Omega$ and Ω is bounded, we have $|x|, |\bar{x}| \le D$, where D is a positive constant that depends on Ω. Also, $|y| < 1$ for $y \in B_1$. Therefore,

$$\begin{aligned} |(-\Delta)^s \mathcal{K}(x)| &\le \frac{C_{N,s}}{2} \int_{B_1} \int_0^1 \frac{|\nabla\mathcal{K}(x+ty)) - \nabla\mathcal{K}(x-ty)|}{|y|^{N+2s-1}}\, dt dy + \frac{2\omega_{N-1}\Lambda}{2s} \\ &\le \frac{C_{N,s}}{2} \int_{B_1} 2t \int_0^1 \frac{(1 + (3t|y| + |\bar{x}|)|x|)}{|y|^{N+2s-2}}\, dt dy + \frac{2\omega_{N-1}\Lambda}{2s} \\ &\le \frac{C_{N,s}}{2} \int_{B_1} \frac{2(1 + (3+D)D)}{|y|^{N+2s-2}}\, dy + \frac{2\omega_{N-1}\Lambda}{2s} \\ &\le C_{N,s}(1 + (3+D)D)\frac{\omega_{N-1}}{2-2s} + \frac{2\omega_{N-1}\Lambda}{2s} \\ &:= M. \end{aligned} \tag{3.5}$$

We denote the constant obtained in the upper bound of $(-\Delta)^s\mathcal{K}(x)$ by M. Observe that for all $x \in \Omega$, we have

$$a(x)v(x) \le a(x)e^{-\theta\mathcal{K}(x)} \le \|a\|_{L^\infty(\Omega)} e^{-\theta\mathcal{K}(x)}.$$

Thus, it follows from (3.2), (3.3) and (3.5) that, for all $x \in \mathbb{R}^N$,

$$Lv(x) \le \\ e^{-\theta\mathcal{K}(x)}\left(\frac{\theta N}{\mathcal{K}(x)} - \theta\frac{q(x)\cdot(x-\bar{x})}{\mathcal{K}(x)} - |x-\bar{x}|^2\Big(\frac{\theta}{\mathcal{K}^3(x)} + \frac{\theta^2}{\mathcal{K}^2(x)}\Big) - \theta M + \|a\|_\infty\right).$$

Now, if $x \in K = B_r(\bar{x}) \cap B_{\frac{r}{2}}(x_0)$, we have $|x-\bar{x}| \ge \frac{r}{2}$ and hence

$$Lv(x) \le \\ e^{-\theta\mathcal{K}(x)}\left(\frac{\theta N}{\mathcal{K}(x)} + \theta\frac{\|q\|_\infty|x-\bar{x}|}{\mathcal{K}(x)} - \frac{r^2}{4}\Big(\frac{\theta}{\mathcal{K}^3(x)} + \frac{\theta^2}{\mathcal{K}^2(x)}\Big) - \theta M + \|a\|_{L^\infty(\Omega)}\right).$$

Since $\mathcal{K}(x) \ge 1$, for all x, we can choose θ large enough so that

$$Lv < 0 \quad \text{in} \quad B_{\frac{r}{2}}(x_0) \cap \Omega.$$

Consider the function $w := -u + \varepsilon v + c_0$, where ε is a positive constant to be chosen later. Since $u \geq c_0$ in $\mathbb{R}^N$, the maximum principle in Theorem 3.1, applied to $u - c_0$, yields that $u > c_0$ in Ω (as $u \not\equiv c_0$ in $\mathbb{R}^N$). As the set $\overline{B}_r(\bar{x}) \setminus K$ is compact, the minimum of u over $\overline{B}_r(\bar{x}) \setminus K$ is attained. So this minimum will be strictly greater than c_0. That is,

$$\min_{\overline{B}_r(\bar{x}) \setminus K} u(x) > c_0.$$

Thus, we can find a constant $\delta > 0$ such that

$$u \geq c_0 + \delta \;\text{ in } \quad \overline{B}_r(\bar{x}) \setminus K.$$

Then, for $x \in \overline{B}_r(\bar{x}) \setminus K$

$$w(x) \leq -\delta + \varepsilon v = -\delta + \varepsilon\Big(e^{-\theta \mathcal{K}(x)} - e^{-\theta(1+r^2)^{\frac{1}{2}}}\Big) \leq -\delta + \varepsilon\Big(1 - e^{-\theta(1+r^2)^{\frac{1}{2}}}\Big).$$

Chosen ε sufficiently small, say

$$\varepsilon < \frac{\delta}{1 - e^{-\theta(1+r^2)^{\frac{1}{2}}}},$$

we have

$$w < 0 \quad \text{in} \quad \overline{B}_r(\bar{x}) \setminus K.$$

Since $u \geq c_0$ and $v < 0$ in $\mathbb{R}^N \setminus B_r(\bar{x})$, we have that $w < 0$ on $\mathbb{R}^N \setminus K$. We also have

$$w < 0 \quad \text{in } \; \partial K \setminus \{x_0\} \;\text{ and }\; w(x_0) = 0.$$

Moreover, using the condition (1.8) on c_0 and $a(x)$, we get

$$Lw = L(-u + \varepsilon v + c_0) \leq c_0 a + \varepsilon Lv < 0 \quad \text{in} \quad K.$$

From the maximum principle, applied to w, we obtain $w < 0$ in K (since $w \not\equiv 0$ in $\mathbb{R}^N$). As $w(x_0) = 0$, it follows that the maximum of w over $\overline{K}$ is attained at x_0. Therefore, the normal derivative $\partial_\nu w$ satisfies

$$\partial_\nu w(x_0) = -\partial_\nu u(x_0) + \varepsilon \partial_\nu v(x_0) \geq 0. \tag{3.6}$$

We now compute the normal derivative of v over $\partial B_r(\bar{x})$. We have

$$\nabla v(x) = -\theta \frac{(x - \bar{x})}{\mathcal{K}(x)} e^{-\theta \mathcal{K}(x)}, \qquad \text{for all } x \in \Omega.$$

Thus, for $x \in \partial B_r(\bar{x})$, we have

$$\partial_\nu v(x) = (x - \bar{x}) \cdot \nabla v(x) = -\theta \frac{|x - \bar{x}|}{\mathcal{K}(x)} e^{-\theta \mathcal{K}(x)} < 0.$$

In particular, $\partial_\nu v(x_0) < 0$ and it follows from (3.6) that

$$\partial_\nu u(x_0) \leq \varepsilon \partial_\nu v(x_0) < 0.$$

This completes the proof. $\square$

We state and prove another version of the Hopf Lemma in the next theorem. We refer the reader to Remark 1.5 above for more details on the difference between Theorem 3.2 and Theorem 1.3.

Theorem 3.2 *(Hopf Lemma). Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain and let q and a be in $L^\infty(\Omega)$ with $a(x) \geq 0$ in Ω. Let $c_0 \in \mathbb{R}$ and $u \in \mathcal{C}^2(\overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^N)$ be such that u is bounded in $\mathbb{R}^N$ and*

$$Lu \geq 0 \text{ in } \quad \Omega. \tag{3.7}$$

Let $x_0 \in \partial\Omega$. Assume that $u(x) = c_0$ on $B_{R_0}(x_0) \cap \partial\Omega$, for some $R_0 > 0$, and that $u \geq c_0$ in $\mathbb{R}^N$. If $u \not\equiv c_0$ in $\mathbb{R}^N$, then

$$\partial_\nu u(x_0) < 0, \tag{3.8}$$

where ν denotes the outer unit normal to $\partial\Omega$ at x_0.

Proof of Theorem 3.2. The proof mostly relies on the fact that

$$(-\Delta)^s u \leq 0 \text{ in } B_\rho(x_0) \cap \Omega, \text{ for some } 0 < \rho < R_0, \tag{3.9}$$

and the Hopf lemma for elliptic operators. The proof of inequality (3.9) is done in details in [9, inequality (2.9) in the proof of Theorem 2.9] and we will omit it here.

We note that if there is a point $y \in B_\rho(x_0)$ such that $u(y) = c_0$, we apply the maximum principle in Theorem 3.1 to (3.7) (knowing that $u \geq c_0$ in $\mathbb{R}^N$) and conclude that $u \equiv c_0$ in $\mathbb{R}^N$, which is a contradiction.

Now, we combine (3.7) and (3.9) to deduce that there exists $\rho > 0$ such that

$$0 \leq Lu \leq -\Delta u + q \cdot \nabla u + au \quad \text{ in } \quad B_\rho(x_0) \cap \Omega, \tag{3.10}$$

and $u \geq c_0$ in $\mathbb{R}^N$. The elliptic maximum principle (see [25, Lemma 3.4], for e.g.) implies that either $u \equiv c_0$ in $B_\rho(x_0) \cap \Omega$ (this cannot happen because of the note above) or $u > c_0$ in $B_\rho(x_0) \cap \Omega$. Moreover, the Hopf Lemma for elliptic operators (here, we have $-\Delta + q \cdot \nabla$) implies that $\partial_\nu u(x) < 0$ for all $x \in B_\rho(x_0) \cap \partial\Omega$, which is part of $\partial(B_\rho(x_0) \cap \Omega)$. In particular, we have $\partial_\nu u(x_0) < 0$ and this completes the proof. $\square$

Next, we give the proof of the interior regularity when $s = 1/2$. We mention that, for $0 < s < 1/2$, we will have regularity $C^{2,\alpha}(\overline{\Omega})$ up to the boundary. The latter is done in Theorem 1.8.

Proof of Theorem 1.6. Let $f \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. Let Ω' and Ω_1 be two open subsets of $\Omega \subset \mathbb{R}^N$ such that

$$\Omega' \subset\subset \Omega_1 \subset\subset \Omega.$$

Define the cut-off function $\eta \in \mathcal{C}_c^\infty(\Omega)$ as

$$\eta(x) = 1 \quad \text{if} \quad x \in \Omega' \qquad \text{and} \qquad \eta(x) = 0 \quad \text{if} \quad x \in \mathbb{R}^N \setminus \Omega_1, \tag{3.11}$$

such that $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}^N$ and there exists $C_1, C_1 > 0$ such that

$$|D\eta| < \frac{C_1}{\operatorname{dist}(\Omega', \partial\Omega)} \qquad \text{and} \qquad |D^2\eta| < \frac{C_2}{(\operatorname{dist}(\Omega', \partial\Omega))^2}.$$

We set

$$v := \eta u \quad \text{and} \quad w := (1-\eta)u$$

Since $u \in W^{2,p}(\Omega)$, it holds that $u \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ applying the Bootstrap argument as in Lemma 4.1 with $\beta = \alpha$. Then, we compute

$$\begin{aligned} -\Delta v &= -\eta\big((-\Delta)^s u + q \cdot \nabla u + au - f\big) - 2\nabla u \cdot \nabla \eta - u\Delta\eta \\ &:= \widetilde{f}. \end{aligned}$$

We note that all elements of $\widetilde{f}$ are supported in Ω_1. We need to show that $\widetilde{f} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$. To do so, we only need to show that $\eta(-\Delta)^s u \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ since the other terms follow easily. We write

$$\eta(-\Delta)^s u = (-\Delta)^s v - u(-\Delta)^s \eta + I(u, \eta)$$

where

$$I(u,\eta)(x) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{(u(x) - u(x+z))(\eta(x) - \eta(x+z))}{|z|^{N+2s}}\, dz.$$

Since $\operatorname{supp} v \subset \Omega$, we have

$$\|v\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^N)} = \|v\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \le C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}. \tag{3.12}$$

We distinguish two cases depending on whether $s = \frac{1}{2}$ or $0 < s < 1/2$.

If $s = \frac{1}{2}$, then, we use the regularity result of [30, Proposition 2.7] with $l = 0$ to get $(-\Delta)^{\frac{1}{2}} v \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ and

$$\|(-\Delta)^{\frac{1}{2}} v\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \le C\|v\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^N)} \le C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}.$$

If $s < 1/2$, then clearly $1+\alpha-2s > \alpha > 0$. Now, if $\alpha < 1+\alpha-2s \leq 1$, we use the regularity result of [30, Proposition 2.7] with $l=0$ to get $(-\Delta)^s v \in \mathcal{C}^{0,1+\alpha-2s}(\mathbb{R}^N)$ and

$$\|(-\Delta)^s v\|_{\mathcal{C}^{0,1+\alpha-2s}(\mathbb{R}^N)} \leq C\|v\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^N)} \leq C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}.$$

Hence, by inclusion of Hölder spaces, we have

$$\|(-\Delta)^s v\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)}\| \leq C\|(-\Delta)^s v\|_{\mathcal{C}^{0,1+\alpha-2s}(\mathbb{R}^N)} \leq C\|v\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^N)} \leq C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}.$$

Now, if $1+\alpha-2s > 1$, then $\alpha-2s > 0$ and [30, Proposition 2.7] with $l=1$ yields that $(-\Delta)^s v \in \mathcal{C}^{1,\alpha-2s}(\mathbb{R}^N)$ and

$$\|(-\Delta)^s v\|_{\mathcal{C}^{1,\alpha-2s}(\mathbb{R}^N)} \leq C\|v\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^N)} \leq C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}.$$

We also know that if $\psi \in \mathcal{C}^{1,\gamma}(\mathbb{R}^N)$ for some $\gamma < \alpha < 1$, then $\psi \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ and

$$\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C\|\psi\|_{\mathcal{C}^{1,\gamma}(\mathbb{R}^N)}. \tag{3.13}$$

Thus, as $0 < \alpha-2s < \alpha < 1$, it follows (taking $\psi = (-\Delta)^s v$ in (3.13)) that

$$\|(-\Delta)^s v\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C\|(-\Delta)^s v\|_{\mathcal{C}^{1,\alpha-2s}(\mathbb{R}^N)} \leq C\|v\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^N)} \leq C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}.$$

The above can be summarized as follows. Given $0 < \alpha < 1$, for all $s \in (0, 1/2]$, the fractional Laplacian $(-\Delta)^s v$ satisfies the estimate

$$\|(-\Delta)^s v\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C\|u\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^N)}, \tag{3.14}$$

where C is a positive constant that depends only on Ω, α, N and s. We also have

$$\|u(-\Delta)^s\eta\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C\|(-\Delta)^s\eta\|_{L^\infty(\mathbb{R}^N)}\|u\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \leq C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}. \tag{3.15}$$

We now show that $I(u,\eta) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$. Note that since Ω_1 is bounded, we let B_{R_0} be a ball centered at zero with radius $R_0 > 0$ and containing Ω_1. We set $R := R_0 + |x| + 1$ for any fixed $x \in \Omega_1$. Observe that if $|z| \geq R$, then $|x+z| \geq |z| - |x| \geq R - |x| = R_0 + 1 > R_0$. Therefore, $\eta(x+z) \equiv 0$ on $\mathbb{R}^N \setminus B_R$. Next, for $x_1, x_2 \in \Omega_1$, we write

$$|I(u,\eta)(x_1) - I(u,\eta)(x_2)| \leq \frac{C_{N,s}}{2}\big(|I_1| + |I_2|\big),$$

where

$$I_1 := \int\limits_{B_R} \frac{\sum_{k=1}^{2}(-1)^{k-1}[(u(x_k)-u(x_k+z))(\eta(x_k)-\eta(x_k+z))]}{|z|^{N+2s}}dz$$

and

$$I_2 := \int\limits_{\mathbb{R}^N\setminus B_R} \frac{\sum_{k=1}^{2}(-1)^{k-1}[(u(x_k)-u(x_k+z))\eta(x_k)]}{|z|^{N+2s}}\,dz$$

We estimate the integrand of I_1 using the fundamental theorem of calculus as follows,

$$\begin{aligned}
&\Big|\sum_{k=1}^{2}(-1)^{k-1}[(u(x_k)-u(x_k+z))(\eta(x_k)-\eta(x_k+z))]\Big| \\
&\qquad = \Big|\int\limits_0^1 \langle \nabla u(x_1+tz)-\nabla u(x_2+tz), z\rangle\,dt\,(\eta(x_1)-\eta(x_1+z)) \\
&\qquad\qquad + \int\limits_0^1 \langle \nabla \eta(x_1+\tau z)-\nabla \eta(x_2+\tau z), z\rangle\,d\tau\,(u(x_2)-u(x_2+z))\Big| \\
&\qquad \le C|z|^2 \int\limits_0^1 |\nabla u(x_1+tz)-\nabla u(x_2+tz)|\,dt \\
&\qquad\qquad + C\|u\|_{\mathcal{C}^1(\overline{\Omega})}|z|^2 \int\limits_0^1 |\nabla \eta(x_1+\tau z)-\nabla \eta(x_2+\tau z)|\,d\tau \\
&\qquad \le C\big(\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}+\|u\|_{\mathcal{C}^1(\overline{\Omega})}\big)|z|^2|x_1-x_2|^\alpha \le C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}|z|^2|x_1-x_2|^\alpha.
\end{aligned}$$

Consequently,

$$|I_1| \le C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}|x_1-x_2|^\alpha \int\limits_{B_R} \frac{|z|^2}{|z|^{N+2s}}\,dz \le C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}|x_1-x_2|^\alpha.$$

We now estimate I_2. Observe first that

$$\begin{aligned}
&\sum_{k=1}^{2}(-1)^{k-1}[(u(x_k)-u(x_k+z))\eta(x_k)] \\
&\qquad = [v(x_1)-v(x_2)]-[u(x_1+z)-u(x_2+z)]\eta(x_1)+[\eta(x_1)-\eta(x_2)]u(x_2+z).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|I_2| &\le C\big(\|v\|_{\mathcal{C}^1(\overline{\Omega})}+\|u\|_{\mathcal{C}^1(\overline{\Omega})}+\|u\|_{L^\infty(\Omega)}\big)|x_1-x_2| \int\limits_{\mathbb{R}^N\setminus B_R} \frac{dz}{|z|^{N+2s}} \\
&\le C\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}|x_1-x_2|.
\end{aligned}$$

Since it not difficult to see that $I(u,\eta)\in L^\infty(\mathbb{R}^N)$, we conclude that $I(u,\eta)\in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$. The latter, together with (3.14) and (3.15), yields that $\widetilde{f}\in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$.

We then consider the equation

$$-\Delta v = \widetilde{f} \quad \text{in} \quad \mathbb{R}^N.$$

Since $\widetilde{f} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$, we can apply the regularity theory for classical elliptic PDEs to see that

$$v \in \mathcal{C}^{2,\alpha}(\overline{\Omega'}).$$

Since $v = u$ in Ω' and since Ω' was arbitrary, the proof of Theorem 1.6 is now complete. □

4. L^p theory and regularity up to the boundary: proof of Theorem 1.7 and Theorem 1.8

This section is dedicated to the L^p-theory of the operator L and to the $\mathcal{C}^{2,\alpha}(\overline{\Omega})$ regularity. We will first prove the following problem

$$\begin{cases} Lu := -\Delta u + (-\Delta)^s u + q \cdot \nabla u + a(x)u &= f \qquad \text{in} \quad \Omega, \\ \quad u = 0 & \qquad \text{on} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.1}$$

has a unique solution a $u \in W^{2,p}(\Omega)$ (see Theorem 1.7). This extends the $W^{2,p}$ estimate done in [31] for L_0 to the case where an advection term and a zero order term are present in the equation. We will need the following lemma.

Lemma 4.1. *Assume that $0 < s < 1/2$, $1 < p < \infty$, the advection $q \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and $a \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ with $a(x) \geq 0$ in Ω. Let $u \in W^{2,p}(\Omega)$ be a solution of*

$$\begin{cases} Lu = 0 & \text{in} \quad \Omega \\ \ u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.2}$$

Then, $u = 0$.

Proof of Lemma 4.1. We need to prove the solution is sufficiently regular first (so that we use the strong maximum principle, stated in Theorem 3.1).

First, consider the case of $p \geq N$ and then note that we have $L_0 u = -q \cdot \nabla u - au$. The right hand side is in $L^T(\Omega)$ for all $T < \infty$ and we can then apply the L^p theory for the operator L_0 in [31] to see that $u \in W^{2,T}(\Omega)$ for all $T < \infty$ and hence $u \in C^{1,\beta}(\overline{\Omega})$ for all $0 < \beta < 1$. Now, since we have $u \in C^{1,\beta}(\overline{\Omega})$, and thanks to the regularity assumption on the boundary of Ω, we can extend u by zero outside Ω and still denote by u (see [25, Lemma 6.37]) and get that the extension is a $C^{0,1}(\mathbb{R}^N)$ function. We can then apply the regularity result of [30, Proposition 2.5] to see that

$$g := (-\Delta)^s u \in C^{0,1-2s}(\mathbb{R}^N).$$

Now, we can write the equation as $-\Delta u = -g - q \cdot \nabla u - au$ in Ω with $u = 0$ on $\partial\Omega$. Hence, as the functions q and a belong to $\mathcal{C}^{0,\alpha}(\overline{\Omega})$, the right hand side $-g - q \cdot \nabla u - au$ is a Hölder function. Thus, $u \in C^{2,1-2s}(\overline{\Omega})$ from the classical theory of elliptic PDEs. We can then apply the maximum principle to get that $u \equiv 0$ in $\mathbb{R}^N$.

Second, we suppose $1 < p < N$ and set $t_1 := \frac{Np}{N-p}$. Then we have

$$u \in W^{2,p}(\Omega) \subset W^{1,t_1}(\Omega)$$

by the Sobolev embedding theorem. Hence, as $L_0 u = -q \cdot \nabla u - au$, the L^p theory for the operator L_0 in [31] yields that $u \in W^{2,t_1}(\Omega)$. Again, the Sobolev embedding theorem implies that $u \in W^{1,t_2}(\Omega)$, where $t_2 := \frac{Nt_1}{N-t_1} > t_1$. If $t_2 < N$, we can do this a finite number of times until we get $u \in W^{2,t}(\Omega)$ for some $t > N$. At this stage, we become in the setting of the first case. The proof of Lemma 4.1 is complete. □

We now have all what is needed to prove Theorem 1.7, which we do as follows.

Proof of Theorem 1.7. We apply the method of continuity. To ease the notation, we define the operator

$$L_0 u := -\Delta u + (-\Delta)^s u,$$

and for $\lambda \in \mathbb{R}$, we consider the family of operators

$$L_\lambda u \equiv (1-\lambda)L_0 u + \lambda L u = L_0 + \lambda q \cdot \nabla u + \lambda a u.$$

Next, for $u \in W^{2,p}(\Omega)$ and $\lambda \in \mathbb{R}$, we consider the problem

$$\begin{cases} L_\lambda u = f & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.3}$$

Let $\mathcal{A}$ be the set given by

$$\mathcal{A} := \left\{ \begin{array}{l} \lambda \in [0,1] : \ \exists C_\lambda > 0 \text{ such that for all } f \in L^p(\Omega), (4.3) \text{ has a} \\ \qquad \text{solution } u \text{ such that } \|u\|_{W^{2,p}(\Omega)} \leq C_\lambda \|f\|_{L^p(\Omega)} \end{array} \right\}. \tag{4.4}$$

In (4.4), we take the constant C_λ to be the smallest constant such that $\|u\|_{W^{2,p}(\Omega)} \leq C_\lambda \|f\|_{L^p(\Omega)}$ holds for all functions $f \in L^p(\Omega)$. In other words, if $C_\lambda > \varepsilon > 0$ then there exists $f_\varepsilon \in L^p(\Omega)$ such that

$$\|u\|_{W^{2,p}(\Omega)} \geq (C_\lambda - \varepsilon)\|f_\varepsilon\|_{L^p(\Omega)}. \tag{4.5}$$

Note that $\mathcal{A}$ is not empty since we have that $0 \in \mathcal{A}$ by [31, Theorem 1.4]. Therefore, we only need to show that $1 \in \mathcal{A}$. To do that, it suffices to prove that $\mathcal{A}$ is both open and closed in [0, 1]. More precisely, it suffices to prove that for any fixed $\lambda_0 \in \mathcal{A}$ and $f \in L^p(\Omega)$, there is an $\varepsilon > 0$ such that $\lambda_0 \pm \varepsilon \in \mathcal{A}$ and that any bounded sequence $\{\lambda_n\}_n \subset \mathcal{A}$ has a convergence subsequence.

$\mathcal{A}$ is open We fix $\lambda_0 \in \mathcal{A}$. We look for a solution $u \in W^{2,p}(\Omega)$ of problem (4.3) in the form $u = v_0 + \Phi$, where v_0 solves (4.3) with $\lambda = \lambda_0$. For $\varepsilon \in \mathbb{R}$, we introduce the operator $\mathcal{N}_\varepsilon$ given by

$$\Psi = \mathcal{N}_\varepsilon(\Phi),$$

where Ψ solves the equation

$$L_{\lambda_0}\Psi = -\varepsilon\left(q\cdot\nabla v_0 + q\cdot\nabla\Phi + a v_0 + a\Phi\right). \tag{4.6}$$

The operator $\mathcal{N}_\varepsilon$ maps $W^{2,p}(\Omega)$ into itself. We claim that if ε is chosen appropriately, then $\mathcal{N}_\varepsilon$ is a contraction in $W^{2,p}(\Omega)$. Indeed, since $\lambda_0 \in A$ there exists a constant $C_{\lambda_0} > 0$ such that

$$\|\mathcal{N}_\varepsilon(\Phi)\|_{W^{2,p}(\Omega)} = \|\Psi\|_{W^{2,p}(\Omega)} \le C_{\lambda_0}\|\varepsilon\left(q\cdot\nabla v_0 + q\cdot\nabla\Phi + a v_0 + a\Phi\right)\|_{L^p(\Omega)}.$$

Now, let Φ_1 and Φ_2 be taken in $W^{2,p}(\Omega)$. Then

$$\begin{aligned}
\|\mathcal{N}_\varepsilon(\Phi_1) - \mathcal{N}_\varepsilon(\Phi_2)\|_{W^{2,p}(\Omega)} &= \|\Psi_1 - \Psi_2\|_{W^{2,p}(\Omega)}\\
&\le |\varepsilon| C_{\lambda_0}\left(\|q\|_{L^\infty(\Omega)}\|\nabla(\Phi_1-\Phi_2)\|_{L^p(\Omega)} + \|a\|_{L^\infty(\Omega)}\|\Phi_1-\Phi_2\|_{L^p}\right)\\
&\le |\varepsilon| C_{\lambda_0} C\left(\|q\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega)}\right)\|\Phi_1-\Phi_2\|_{W^{2,p}(\Omega)},
\end{aligned}$$

where the constant C and C_{λ_0} are independent of ε and Φ. Taking ε such that

$$|\varepsilon| \le \frac{1}{2C_{\lambda_0}C\left(\|q\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega)}\right)},$$

we get that $\mathcal{N}_\varepsilon$ is a contraction mapping. By the fixed point theorem, for each such ε there exists a fixed point Φ such that $\mathcal{N}_\varepsilon(\Phi) = \Phi$. We just showed that the equation $L_{\lambda_0\pm\varepsilon}u = f$ has a solution $u \in W^{2,p}(\Omega)$. Moreover, as $\lambda_0 \in \mathcal{A}$, it follows from (4.6) that for ε small enough we have

$$\begin{aligned}
\|\Phi\|_{W^{2,p}(\Omega)} &= \|\Psi\|_{W^{2,p}(\Omega)} = \|\mathcal{N}_\varepsilon(\Phi)\|_{W^{2,p}(\Omega)}\\
&\le |\varepsilon| C_{\lambda_0} C\left(\|q\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega)}\right)\left(\|\Phi\|_{W^{2,p}(\Omega)} + \|v_0\|_{W^{2,p}(\Omega)}\right)\\
&\le \frac{1}{2}\left(\|\Phi\|_{W^{2,p}(\Omega)} + \|v_0\|_{W^{2,p}(\Omega)}\right).
\end{aligned}$$

This means that $\|\Phi\|_{W^{2,p}(\Omega)} \le 2\|v_0\|_{W^{2,p}(\Omega)}$ and the norm of u in $W^{2,p}(\Omega)$ becomes

$$\begin{aligned}
\|u\|_{W^{2,p}(\Omega)} &\le \left(\|v_0\|_{W^{2,p}(\Omega)} + \|\Phi\|_{W^{2,p}(\Omega)}\right)\\
&\le C_2\|v_0\|_{W^{2,p}(\Omega)}\\
&\le C\|f\|_{L^p(\Omega)}.
\end{aligned}$$

Therefore, $\lambda_0 \pm \varepsilon \in \mathcal{A}$ and this proves that $\mathcal{A}$ is open.

$\mathcal{A}$ is closed In order to complete the proof of theorem, we show that $\mathcal{A}$ is closed. Let then $\{\lambda_n\}_n \subset \mathcal{A}$ be a sequence such that $\lambda_n \to \lambda_0 \in \mathbb{R}$ as $n \to \infty$. We claim that $\lambda_0 \in \mathcal{A}$. Let $f \in L^p(\Omega)$. Since $\lambda_n \in \mathcal{A}$ for any n, there exists u_n that satisfies (4.3) with λ_n in place of λ and

$$\|u_n\|_{W^{2,p}(\Omega)} \le C_n\|f\|_{L^p(\Omega)}$$

where $C_n := C_{\lambda_n}$.

If the sequence $\{C_n\}_n$ is bounded, then the sequence $\{u_n\}_n$ is uniformly bounded in $W^{2,p}(\Omega)$. Thus, passing to a subsequence, we have

$$u_n \rightharpoonup u \;\text{ weakly in }\; W^{2,p}(\Omega) \quad \text{and} \quad u_n \to u \;\text{ strongly in }\; W^{1,p}(\Omega) \text{ as } n \to +\infty.$$

The function u solves problem (4.3) with λ_0 in place of λ and

$$\|u\|_{W^{2,p}(\Omega)} \le \liminf_{n\to\infty} \|u_n\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

This shows that $\lambda_0 \in \mathcal{A}$ and ends the proof in the case where $\{C_n\}_n$ is bounded.

Indeed, we will show next that this is the only possible case (i.e. $\{C_n\}_n$ is bounded). Assume to the contrary that $\{C_n\}_n$ is unbounded. Then, passing to a subsequence, we may have that $C_n \to \infty$ as $n \to \infty$. Thus, there exists a sequence $\{f_n\}_n$ such that for large n we have

$$\|u_n\|_{W^{2,p}(\Omega)} \ge (C_n - 1)\|f_n\|_{L^p(\Omega)}.$$

Note that the above inequality holds because of (4.5), in which the constant $C_n := C_{\lambda_n}$ is the smallest constant such that $\|u\|_{W^{2,p}(\Omega)} \le C_n\|f\|_{L^p(\Omega)}$ holds. We let

$$t_n := \|u_n\|_{W^{2,p}(\Omega)}, \quad v_n := \frac{u_n}{t_n} \quad \text{and} \quad \widetilde{f_n} := \frac{f_n}{t_n}.$$

Then $\|v_n\|_{W^{2,p}(\Omega)} = 1$ for all n, $\widetilde{f_n} \to 0$ in $L^p(\Omega)$ as $n \to \infty$, and v_n satisfies the equation

$$L_0 v_n = -\lambda_n q \cdot \nabla v_n - \lambda_n a v_n + \widetilde{f_n} \quad \text{in} \quad \Omega, \qquad v_n = 0 \;\text{ on }\; \mathbb{R}^N \setminus \Omega. \tag{4.7}$$

Let $\bar{\lambda} > 0$ be such that $|\lambda_n| \le \bar{\lambda}$ for all n (recall that $\lambda_n \to \lambda_0$) and $K > 0$ such that $\|\widetilde{f_n}\|_{L^p(\Omega)} \le K$ for all n. We now apply [31, Theorem 1.4] to the operator L_0 appearing in (4.7) and conclude that there exists $C > 0$ (depending only on Ω) such that, for all n, we have

$$\begin{aligned}
\|v_n\|_{W^{2,p}(\Omega)} &\le C\left(\|\widetilde{f_n} - \lambda_n q \cdot \nabla v_n - \lambda_n a v_n\|_{L^p(\Omega)}\right)\\
&\le C\left(\|\widetilde{f_n}\|_{L^p(\Omega)} + \bar{\lambda}\|q\|_{L^\infty(\Omega)}\|v_n\|_{W^{1,p}(\Omega)} + \bar{\lambda}\|a\|_{L^\infty(\Omega)}\|v_n\|_{L^p(\Omega)}\right)\\
&\le C\left(K + \bar{\lambda}\|q\|_{L^\infty(\Omega)} + \bar{\lambda}\|a\|_{L^\infty(\Omega)}\right).
\end{aligned}$$

Thus the sequence $\{v_n\}_n$ is uniformly bounded in $W^{2,p}(\Omega)$. The Banach-Alaoglu Theorem then implies the existence of a subsequence, which we still label as $\{v_n\}_n$, that converges weakly to some $v_0 \in W^{2,p}(\Omega)$ and strongly in $W^{1,p}(\Omega)$ thanks to the compactness of the Sobolev embedding. Hence $\|v_0\|_{W^{2,p}(\Omega)} \le 1$ (as $\|v_0\| \le \liminf_{n\to\infty} \|v_n\|_{W^{2,p}(\Omega)}$). We now consider two possibilities.

If $v_0 = 0$, this will contradict the normalization $\|v_n\|_{W^{2,p}(\Omega)} = 1$: indeed, if $v_0 = 0$, then

$$\lambda_n q \cdot \nabla v_n \to 0 \text{ and } a(x)v_n \to 0 \quad \text{strongly in } L^p(\Omega)$$

as we have strong convergence of $\{v_n\}_n$ in $W^{1,p}$ and the functions q and a are bounded on $\overline{\Omega}$. Also, $\tilde{f}_n \to 0$ strongly in $L^p(\Omega)$. Hence, we can use the L^p theory for L_0 ([31, Theorem 1.4]) and (4.7) to see that $v_n \to 0$ in $W^{2,p}(\Omega)$, which contradicts the normalization of v_n in $W^{2,p}(\Omega)$.

The other possibility is that $v_0 \neq 0$. In such case, we can pass to the limit in (4.7) to get

$$L_{\lambda_0} v_0 = 0 \quad \text{in} \quad \Omega, \qquad v_0 = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \tag{4.8}$$

We will discuss the consequences of (4.8) in what follows.

The interior regularity result, in Theorem 1.6, allows us to conclude that $v_0 \in \mathcal{C}^2(\Omega)$. The Sobolev embedding also tells us that $v_0 \in C^{1,\alpha}(\overline{\Omega})$. Thus, $v_0 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\mathbb{R}^N)$ (after extending v_0 by 0 on $\mathbb{R}^N \setminus \Omega$). The maximum principle in Theorem 3.1 implies that $v_0 \equiv 0$, which is a contradiction.

Thus, for $s \in (0, 1/2]$, the assumption that $\{C_n\}_n$ is unbounded leads to a contradiction. Hence $\{C_n\}_n$ is bounded and this in turn yields that $\lambda_0 \in \mathcal{A}$, as we showed earlier. Therefore, $\mathcal{A}$ is closed and this completes the proof of Theorem 1.7. □

Proof of Theorem 1.8. We use the L^p theory derived in Theorem 1.7. Indeed, since $f \in C^{0,\alpha}(\overline{\Omega})$ and Ω is bounded, $f \in L^p(\Omega)$ for any $p < \infty$. It then follows from Theorem 1.7 that $u \in C^{1,\beta}(\overline{\Omega})$ for any $\beta \in (0, 1)$ (we choose all $p > N$). Take in particular $\beta = \alpha$. From the uniqueness of the solution, we get that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \lesssim \|f\|_{L^p(\Omega)} \leq C_1 \|f\|_{C^{0,\alpha}(\overline{\Omega})}, \tag{4.9}$$

for some constant $C_1 > 0$.

As before, we can extend u by zero outside Ω by a $C^{0,1}$ function in $\mathbb{R}^N$ and still denote the extension by u. We apply again the regularity result of [30, Proposition 2.5] to see that $g := (-\Delta)^s u \in C^{0,1-2s}(\mathbb{R}^N)$ with a control on the $\mathcal{C}^{0,1-2s}$- norm of g as follows

$$\|g\|_{C^{0,1-2s}(\mathbb{R}^N)} \lesssim \|u\|_{C^{0,1}(\mathbb{R}^N)} \leq \|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_1 \|f\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}.cc \tag{4.10}$$

As $\alpha < 1 - 2s$, we get

$$\|g\|_{C^{0,\alpha}(\mathbb{R}^N)} \lesssim \|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_1 \|f\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}. \tag{4.11}$$

Next, since q and a are Hölder over $\overline{\Omega}$, we have that $q \cdot \nabla u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and $a(x)u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. Now, we write the equation as

$$-\Delta u = -g - q \cdot \nabla u - au \text{ in } \Omega \quad \text{with } u = 0 \quad \text{on } \partial\Omega.$$

Hence, as Ω has smooth boundary, it follows from the classical theory of elliptic PDEs that $u \in C^{2,\alpha}(\overline{\Omega})$ and there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
\|u\|_{C^{2,\alpha}(\overline{\Omega})} &\leq C_2 \|-g - q \cdot \nabla u - au\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\leq C_3 \left(\|g\|_{C^{0,\alpha}(\mathbb{R}^N)} + \|q\|_{C^{0,\alpha}(\overline{\Omega})} \|u\|_{C^{1,\alpha}(\overline{\Omega})} + \|a\|_{C^{0,\alpha}(\overline{\Omega})} \|u\|_{C^{0,\alpha}(\overline{\Omega})} \right).
\end{aligned}$$

We now combine the latter estimate with (4.9) and (4.11) to conclude there exists a constant $C > 0$ such that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{0,\alpha}(\overline{\Omega})}.$$

The proof of Theorem 1.8 is now complete. $\square$

5. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2. We recall the following statement of Krein-Rutman Theorem from [17, Theorem 1.2] as we will use it in the proof of Theorem 1.2.

Theorem A *(Krein-Rutman Theorem, [17]). Let X be a Banach space, $K \subset X$ a solid cone (that is, K has a nonempty interior), and let $T : X \to X$ a compact linear operator which satisfies $T(K \setminus \{0\}) \subset K^\circ$ (K° denotes the interior of K). Then,*

- *(i) $r(T) > 0$ and $r(T)$ is a simple eigenvalue with an eigenfunction $v \in K^\circ$; there is no other eigenvalue with a positive eigenfunction.*
- *(ii) $|\mu| < r(T)$ for all eigenvalues $\mu \neq r(T)$.*

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2-(a). We define the space

$$X := \left\{u \in C^{0,1}(\overline{\Omega}) : \quad u = 0 \text{ on } \partial\Omega \text{ and } u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\right\}$$

and the cone

$$K := \left\{u \in X : \quad u \geq 0 \text{ in } \overline{\Omega}\right\}.$$

We will denote the interior of K by K°. Indeed,

$$K^\circ = \{u \in X : \text{ there is some } \varepsilon > 0 \text{ such that } u(x) \geq \varepsilon \mathrm{dist}(x, \partial\Omega)) \text{ for all } x \in \Omega\}.$$

It is important here to highlight that the coefficient $a(x)$ is not assumed to be nonnegative. We let $M > 0$ be a large enough positive constant so that

$$\widetilde{a}(x) := a(x) + M \geq 0 \text{ for all } x \in \overline{\Omega}. \tag{5.1}$$

We now define the operator

$$T : X \to X$$

by $Tf = u$, where u is the solution of the problem

$$\begin{cases} Lu(x) + Mu(x) := -\Delta u(x) + (-\Delta)^s u(x) + q \cdot \nabla u + \widetilde{a}(x)u(x) = f \text{ in } \Omega \\ \qquad\qquad u = 0 \text{ on } \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.2}$$

Problem (5.2) admits a unique solution $u \in W^{2,p}(\Omega)$ for all $p \geq 1$, because of (5.1), the fact that $f \in \mathcal{C}^{0,1}(\overline{\Omega}) \subset \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and Theorem 1.7. The operator T is linear. Moreover, T is bounded since, by Theorem 1.7 and the Sobolev embedding, we have $\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leq C\|f\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}$. Thus,

$$\|u\|_{\mathcal{C}^{0,1}(\overline{\Omega})} \lesssim \|f\|_{\mathcal{C}^{0,1}(\overline{\Omega})}.$$

Let us now prove that $T(K \setminus \{0\}) \subseteq K^\circ$. Let $f \in K$ such that $f \not\equiv 0$ and set $Tf = w$. Hence, $Lw + Mw = f$ and w satisfies (5.2).
The interior regularity of w, that is $w \in \mathcal{C}^2(\Omega)$, follows from Theorem 1.6 as w solves (5.2) and $\widetilde{a} \geq 0$ in Ω. Moreover, $w \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ by the Sobolev embedding ($w \in W^{2,p}(\Omega)$ for a large enough $p > N$). Thus, after extending w by 0 on $\mathbb{R}^N \setminus \Omega$ (we still denote the extension by w), we have $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\alpha}(\overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^N)$. Applying the Hopf Lemma—stated in Theorem 1.3, we obtain that $\partial_\nu w(x) < 0$ for all $x \in \partial\Omega$.

Thus, for all $s \in (0, 1/2]$, we have $\partial_\nu w(x) < 0$ for all $x \in \partial\Omega$. As $\partial\Omega$ is compact, then $\max\limits_{\partial\Omega} \partial_\nu w(x) < 0$. This allows us to find an open $\mathcal{C}^{0,1}$ neighborhood $\mathcal{O}$ of w, such that

$$\mathcal{O} \subseteq \{u \in X : \text{ there is some } \varepsilon > 0 \text{ such that } u(x) \geq \varepsilon \mathrm{dist}(x, \partial\Omega)), \forall x \in \Omega\} \subseteq K^\circ.$$

Thus, $w = Tf \in K^\circ$.

We now verify that T is compact. Let $\{f_n\}_n \subset X$ be a bounded sequence in X. Let us say that $\|f_n\|_{L^\infty(\Omega)} \leq 1$. It follows that $f_n \in L^p(\Omega)$ for any $1 < p < \infty$ and from the L^p- theory in Theorem 1.7, we have that $u_n := Tf_n \in W^{2,p}(\Omega)$ for any $1 < p < \infty$. The Sobolev embedding implies that $u_n = Tf_n \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$. Furthermore, the estimate (1.13) coupled with the Sobolev embedding now reads

$$\|u_n\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \lesssim \|u_n\|_{W^{2,p}(\Omega)} \lesssim \|f_n\|_{L^p(\Omega)} \leq C,$$

where C is a constant independent of n. This implies that $\{Tf_n\}_n$ is bounded in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$. By the Arzela-Ascoli theorem, the sequence $\{Tf_n\}_n$ has a convergent subsequence (the convergence of the subsequence holds in $\mathcal{C}^1(\overline{\Omega})$ and hence in $\mathcal{C}^{0,1}(\overline{\Omega})$). This proves that T is compact.

Therefore, we can apply the Krein-Rutman theorem to assert that there exists a unique positive real number $\varrho(T) > 0$ and a unique (up to multiplication by a nonzero constant) positive function $f \in K^\circ$ such that $Tf = \varrho(T)f$. Therefore, the function $\varphi_1 := Tf > 0$ in Ω satisfies the problem

$$\begin{cases} L\varphi_1 + M\varphi_1 = (1/\varrho(T))\varphi_1 & \text{in} \quad \Omega \\ \qquad\qquad \varphi_1 = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.3}$$

Furthermore, as $\varphi_1 = Tf$ and $f \in K^\circ$, it follows that $\varphi_1 \in K^\circ$ and hence

$$\partial_\nu \varphi_1(x) < 0 \text{ for all } x \in \partial\Omega. \tag{5.4}$$

We set $\kappa(\Omega, q) := 1/\varrho(T) > 0$ and let

$$\lambda_1(\Omega, q) := \kappa(\Omega, q) - M. \tag{5.5}$$

Then φ_1 satisfies

$$\begin{cases} L\varphi_1 = \lambda_1(\Omega, q)\varphi_1 & \text{in} \quad \Omega \\ \varphi_1 = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.6}$$

To see that φ_1 is the principal eigenfunction of L, we show that any function φ satisfying (5.6) must be a constant multiple of φ_1. Indeed, let φ be such that

$$L\varphi = \lambda_1(\Omega, q)\varphi \quad \text{in} \quad \Omega$$

and $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$. Then,

$$L\varphi + M\varphi = (\lambda_1(\Omega, q) + M)\varphi = \kappa(\Omega, q)\varphi \quad \text{in} \quad \Omega.$$

Dividing by $\kappa(\Omega, q) > 0$, we then have

$$(L + M\,\mathrm{Id})\frac{\varphi}{\kappa(\Omega, q)} = \varphi \quad \text{in} \quad \Omega.$$

The latter implies that $T\varphi = \varrho(T)\varphi$ in Ω with $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$. In other words, φ is an eigenfunction of T associated with $\varrho(T)$. The fact that $\varrho(T)$ is the principal eigenvalue of T implies that $\varphi = c\varphi_1$ for some constant $c \neq 0$. Therefore, φ_1 is the principal eigenfunction of L and hence $\lambda_1(\Omega, q)$ is the principal eigenvalue of L. The proof of part **(a)** is now complete.

Proof of (b). We first recall that the function φ_1 was introduced in the proof of **(a)** as $\varphi_1 = Tf$ where $f \in K^\circ \subset \mathcal{C}^{0,1}(\overline{\Omega})$. Hence, $\varphi_1 \in L^p(\Omega)$ for all $p > 1$. Looking at (5.3), and applying Theorem 1.7 (note that $a(x) + M \geq 0$), we get $\varphi_1 \in W^{2,p}(\Omega)$ for all $p > 1$. Thus, choosing p large enough we have $\varphi_1 \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$. As $\varphi_1 \in \mathcal{C}^{0,1}(\overline{\Omega})$, we can extend φ_1 to a continuous function over the whole space $\mathbb{R}^N$ and keep $\varphi_1 = 0$ on $\mathbb{R}^N \setminus \Omega$. So $\varphi_1 \in \mathcal{C}^{1,\alpha}(\overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^N)$. We now apply Theorem 1.6 to (5.3) and get $\varphi_1 \in \mathcal{C}^{2,\alpha}(\overline{\Omega'})$ for all $\Omega' \subset\subset \Omega$. Hence $\varphi_1 \in \mathcal{C}^2(\Omega)$. In summary,

$$\varphi_1 \in \mathcal{C}^{1,\alpha}(\overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^N) \cap \mathcal{C}^2(\Omega) \text{ whenever } 0 < s \leq 1/2 \text{ and } 0 < \alpha < 1.$$

If we assume further that $0 < s < 1/2$ and $0 < \alpha < 1 - 2s$, then Theorem 1.8 yields that $\varphi_1 \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. This completes the proof of **(b)**.

Proof of (c). Here, we have a *real eigenvalue* $\lambda \neq \lambda_1(\Omega, q)$ of L and we want to show that $\lambda > \lambda_1(\Omega, q)$. We let ψ be the eigenfunction of L associated with λ. Then ψ must be real-valued since $\lambda \in \mathbb{R}$ and the coefficients q, a and f are real-valued functions on Ω. We then have

$$\begin{cases} (L + M)\psi = (\lambda + M)\psi & \text{in} \quad \Omega \\ \psi = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.7}$$

and φ_1 satisfies (5.6). Recall that if $\lambda = \lambda_1(\Omega, q)$, then we must have $\psi \in \mathbb{R}\varphi_1$.

Assume to the contrary that $\lambda \leq \lambda_1(\Omega, q)$. As we are given that $\lambda \neq \lambda_1(\Omega, q)$, our assumption becomes that $\lambda < \lambda_1(\Omega, q)$. Without loss of generality, as φ_1 has a constant sign on Ω, we may assume that $\varphi_1 > 0$ in Ω and $\varphi_1 = 0$ on $\mathbb{R}^N \setminus \Omega$. Then, by (5.4) we have $\partial_\nu \varphi_1 < 0$ on $\partial\Omega$ and so we can find $t_0 > 0$ such that

$$\forall t \in \mathbb{R} \text{ with } |t| \leq t_0, \text{ we have } \varphi_1 \geq t\psi \quad \text{in} \quad \Omega.$$

We now let

$$t^* := \sup\{t \geq 0 : \quad \varphi_1 \geq t\psi \quad \text{ in } \Omega\}. \tag{5.8}$$

It is clear that $t^* \geq t_0 > 0$ and t^* is finite since the functions ψ and φ_1 are in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ (hence, bounded functions on $\overline{\Omega}$). Also, we have $\varphi_1 \geq t^*\psi$ in Ω. We denote

$$z := \varphi_1 - t^*\psi$$

and see that z satisfies

$$\begin{aligned} \forall x \in \Omega, \quad (L+M)z(x) &= (L+M)\varphi_1(x) - t^*(L+M)\psi(x) \\ &= (\lambda_1(\Omega,q)+M)\varphi_1(x) - (\lambda+M)t^*\psi(x) \\ &> (\lambda+M)z(x), \text{ since } \lambda_1(\Omega,q) > \lambda. \end{aligned}$$

As $z \geq 0$ in Ω, we may still add a large enough constant $\tilde{M} > 0$, so that $\lambda + M + \tilde{M} \geq 0$, and have $(\lambda + M + \tilde{M})z \geq 0$ in Ω. Thus, z satisfies

$$\begin{cases} (L+M+\tilde{M})z \geq 0 & \text{in} \quad \Omega \\ z = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \\ z \geq 0 & \text{in} \quad \Omega. \end{cases} \tag{5.9}$$

Two alternatives may then occur.
Alternative 1. There exists $x_0 \in \Omega$ such that $z(x_0) = 0$. In such case, as

$$a(x) + M + \tilde{M} \geq 0 \text{ in } \Omega,$$

the maximum principle (Theorem 3.1) yields that $z \equiv 0$ in $\mathbb{R}^N$. Hence, $\varphi_1 = t^*\psi$ and, by the uniqueness of φ_1 (as a principal eigenfunction), we must then have $\lambda = \lambda_1(\Omega, q)$. This contradicts our assumption that $\lambda < \lambda_1(\Omega, q)$.
Alternative 2. As Alternative 1 is ruled out, we will have $z > 0$ in Ω. Then, $\varphi_1 > t^*\psi$ in Ω and, because $\partial_\nu\varphi_1 < 0$ on $\partial\Omega$ (see (5.4)), we can still find $\delta > 0$ such that $\varphi_1 \geq (t^* + \delta)\psi$ in Ω. However, this contradicts the definition of t^* in (5.8).
In both cases, we obtained a contradiction. Therefore, our assumption is false and we must have $\lambda \geq \lambda_1(\Omega, q)$ for all real eigenvalues of L. The proof of **(c)** is now complete.

Proof of (d). From part (ii) of Theorem A, we know that any eigenvalue $\mu \neq \varrho(T)$ of T satisfies $|\mu| < \varrho(T)$. This means that any eigenvalue $\mu \neq 1/\varrho(T)$ of $L + M\mathrm{Id}$ satisfies

$$|\mu| > \frac{1}{\varrho(T)} := \kappa(\Omega, q).$$

Now, let $\lambda \in \mathbb{C}$ be an eigenvalue of L such $\lambda \neq \lambda_1(\Omega, q)$ and let ψ be an eigenfunction associated with λ. Then, $\lambda + M$ is an eigenvalue of $L + M\,\mathrm{Id}$ and ψ is still an eigenfunction associated with $\lambda + M$. Hence, $|\lambda + M| > 1/\varrho(T)$. From (5.5), it follows that

$$|\lambda + M| - M > \frac{1}{\varrho(T)} - M = \lambda_1(\Omega, q).$$

Hence, $|\lambda| > \lambda_1(\Omega, q)$.

In what follows, we prove the stronger inequality, $\Re(\lambda) \geq \lambda_1(\Omega, q)$, for any eigenvalue (possibly complex) λ of L. Since the coefficients a, q and f of L are real-valued functions, it follows that if the eigenfunction ψ associated with λ is real-valued, then λ must be a real number. In such case, the claim in (d) follows from (c).

It remains to check the case where the eigenfunction ψ is complex-valued. We will use the same idea of [7] with the distinction that our operator here involves a fractional Laplacian. In this case, we let $\widehat{\Omega} \subset \mathbb{R}^N \times \mathbb{R}^N$ be the open domain

$$\widehat{\Omega} := \Omega \times \Omega.$$

A point in $\widehat{\Omega}$ is then denoted by (x, y), where $x, y \in \Omega$. We denote the action of L in the x variables by L_x and the action of L in the y variables by L_y and set[1]

$$\widehat{L} = L_x + L_y.$$

Taking $\widehat{\varphi_1}(x, y) := \varphi_1(x)\varphi_1(y)$, we note that

$$\widehat{L}\widehat{\varphi_1}(x, y) = 2\lambda_1(\Omega, q)\varphi_1(x)\varphi_1(y) = 2\lambda_1(\Omega, q)\widehat{\varphi_1}(x, y),$$

for all $(x, y) \in \widehat{\Omega}$ and $\widehat{\varphi_1}(x, y) = 0$ for all $(x, y) \in (\mathbb{R}^N \times \mathbb{R}^N) \setminus \widehat{\Omega}$. Thus, $2\lambda_1(\Omega, q)$ is an eigenvalue of $\widehat{L}$ when considered over the domain $\widehat{\Omega}$. Since $\widehat{\varphi_1} > 0$ in $\widehat{\Omega}$, it follows from part (a) of this theorem (applied to $\widehat{L}$) that $2\lambda_1(\Omega, q)$ is the principal eigenvalue of $\widehat{L}$. Now, we define the function $\widehat{\psi}$ on $\widehat{\Omega}$ as

$$\widehat{\psi}(x, y) := \psi(x)\overline{\psi(y)} + \overline{\psi(x)}\psi(y), \quad \text{for all } \ (x, y) \in \widehat{\Omega}.$$

Observe that $\widehat{\psi}$ is a real-valued function. Since $L\psi = \lambda\psi$ and $\psi = 0$ on $\mathbb{R}^N \setminus \Omega$, it follows that $\widehat{\psi}$ satisfies

$$\forall (x, y) \in \widehat{\Omega}, \quad \widehat{L}\widehat{\psi}(x, y) = 2(\lambda + \bar{\lambda})\widehat{\psi}(x, y),$$

with

$$\widehat{\psi} = 0 \quad \text{on} \quad (\mathbb{R}^N \times \mathbb{R}^N) \setminus \widehat{\Omega}.$$

[1] For a function $U : \widehat{\Omega} = \Omega \times \Omega \to \mathbb{C}$, we have

$$\begin{aligned}\forall (x, y) \in \widehat{\Omega}, \ \widehat{L}U(x, y) &= L_x U(x, y) + L_y U(x, y) \\ &= -\Delta_x U(x, y) + (-\Delta_x)^s U(x, y) + q(x) \cdot \nabla_x U(x, y) + a(x)U(x, y) \\ &\quad - \Delta_y U(x, y) + (-\Delta_y)^s U(x, y) + q(y) \cdot \nabla_y U(x, y) + a(y)U(x, y).\end{aligned}$$

By $\Delta_x U(x, y)$, we mean $\sum_{i=1}^N \partial_{x_i x_i} U(x_1, \cdots, x_N, y)$.

Thus, $2\Re(\lambda) = \lambda + \bar{\lambda}$ is a *real eigenvalue* of $\widehat{L}$ and it is associated with the *real-valued eigenfunction* $\widehat{\psi}$. Applying part **(c)** to $\widehat{L}$, we conclude that $2(\lambda + \bar{\lambda})$ must be greater or equal to the principal eigenvalue of $\widehat{L}$. That is, $2(\lambda + \bar{\lambda}) \geq 2\lambda_1$ and hence $\Re(\lambda) \geq \lambda_1$.

Next, we prove the strict inequality $\Re(\lambda) > \lambda_1(\Omega, q)$ under the assumption that $\lambda \neq \lambda_1(\Omega, q)$. Suppose to the contrary that $\Re(\lambda) = \lambda_1(\Omega, q)$ and so $2(\lambda + \bar{\lambda}) = 2\lambda_1(\Omega, q)$. Since $2(\lambda + \bar{\lambda}) = 2\lambda_1(\Omega, q)$ is the principal eigenvalue of $\widehat{L}$, then we can apply part **(a)** of this theorem to $\widehat{L}$ and conclude that the (real) eigenfunction $\widehat{\psi} \in \mathbb{R}\widehat{\varphi_1}$. Thus, there exists a real constant $c \neq 0$ such that

$$\forall x, y \in \Omega, \quad \psi(x)\overline{\psi(y)} + \overline{\psi(x)}\psi(y) = c\varphi_1(x)\varphi_1(y). \tag{5.10}$$

Taking $x = y$ in (5.10) implies that $2|\psi(x)|^2 = c(\varphi_1(x))^2$ for all $x \in \Omega$. As $\varphi_1 > 0$ in Ω, it follows that $c > 0$ and $|\psi(x)| = \sqrt{c/2}\varphi_1(x)$ for all $x \in \Omega$. So we can write the complex-valued eigenfunction ψ as

$$\psi(x) = \sqrt{c/2}\varphi_1(x)e^{i\theta(x)} \text{ for all } x \in \Omega, \tag{5.11}$$

where $\theta(x)$ is a real-valued function defined on Ω. Implementing (5.11) in (5.10), we obtain

$$\cos(\theta(x) - \theta(y)) = 1 \text{ for all } (x, y) \in \Omega \times \Omega.$$

Thus, θ must be identically constant over Ω (say $\theta \equiv \theta_0$, for some $\theta_0 \in \mathbb{R}$). If $\theta_0 = 0$, then (5.11) yields $\psi \in \mathbb{R}\varphi_1$ and consequently $\lambda = \lambda_1(\Omega, q)$, which is a contradiction. Thus, $\theta_0 \neq 0$ and we now rewrite (5.11), with $\theta \equiv \theta_0$, as

$$\psi(x) = A\varphi_1(x) \quad \text{for all } \ x \in \Omega, \tag{5.12}$$

where $A = \sqrt{c/2}e^{i\theta_0} \in \mathbb{C} \setminus \mathbb{R}$. It follows that

$$L\psi = \lambda\psi = \lambda A\varphi_1 \quad \text{in} \quad \Omega$$

and, on the other hand,

$$L\psi = L(A\varphi_1) = AL\varphi_1 = \lambda_1(\Omega, q)A\varphi_1 \quad \text{in} \quad \Omega.$$

As $\varphi_1 > 0$ in Ω and $A \neq 0$, it follows that $\lambda = \lambda_1(\Omega, q)$ and this is again a contradiction. Therefore, $\Re(\lambda) > \lambda_1(\Omega, q)$ and this completes the proof of assertion **(d)** in Theorem 1.2.

Proof of (e). We are left to prove the max-inf formulation (1.7) of $\lambda_1(\Omega, q)$, which is stated in part **(e)** of Theorem 1.2. We recall that $\mathcal{V}(\Omega)$ is given by

$$\mathcal{V}(\Omega) := \Big\{u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_c(\mathbb{R}^N) : \ u > 0 \text{ in } \Omega \ \text{ and } \ u \equiv 0 \text{ on } \ \mathbb{R}^N \setminus \Omega\Big\}.$$

Since $\varphi_1 \in \mathcal{V}(\Omega)$, it follows that

$$\lambda_1(\Omega, q) \leq \sup_{u \in \mathcal{V}(\Omega)} \inf_{x \in \Omega} \frac{Lu(x)}{u(x)}.$$

Thus, we only need to prove that

$$\lambda_1(\Omega, q) \geq \sup_{u \in \mathcal{V}(\Omega)} \inf_{x \in \Omega} \frac{Lu(x)}{u(x)}.$$

Once we have equality then we can replace the sup with a max. So we now argue by contradiction. Suppose that

$$\lambda_1(\Omega, q) < \sup_{u \in \mathcal{V}(\Omega)} \inf_{x \in \Omega} \frac{Lu(x)}{u(x)}.$$

Then, there exists $\varepsilon > 0$ and a function $v \in \mathcal{V}(\Omega)$ such that

$$\lambda_1(\Omega, q) + \varepsilon < \inf_{x \in \Omega} \frac{Lv(x)}{v(x)}. \tag{5.13}$$

Then note we have

$$Lv > (\lambda_1(\Omega, q) + \varepsilon)v \text{ in } \Omega \ \text{ with } v = 0 \ \text{ on } \ \mathbb{R}^N \backslash \Omega.$$

Taking $M > 0$ so that (5.1) holds, and thanks to the fact that $\lambda_1(\Omega, q) + M > 0$, we have

$$Lv + Mv > (\lambda_1(\Omega, q) + \varepsilon + M)v > 0 \ \text{ in } \ \Omega, \qquad \text{with } v = 0 \text{ on } \mathbb{R}^N \backslash \Omega,$$

and hence by Hopf's Lemma (Theorem 1.3) we have $\partial_\nu v < 0$ on $\partial\Omega$.

We now define

$$\tau^* := \sup\{\tau > 0 : \ v - \tau\varphi_1 \geq 0 \ \text{ in } \ \Omega\} \tag{5.14}$$

and note $0 < \tau^* < \infty$ since φ_1 is sufficiently regular and the above Hopf result for v.

We now set $w = v - \tau^*\varphi_1$. First note that $w \geq 0$ in Ω and $w = 0$ on $\mathbb{R}^N \setminus \Omega$. Since $\varepsilon > 0$ we see that v cannot be a multiple of φ_1 and hence w is not identically zero. Then note we have

$$Lw = Lv - \tau^* L\varphi_1 > \varepsilon v + \lambda_1(\Omega, q)w \geq 0 \quad \text{ in } \Omega.$$

Hence,

$$Lw + Mw = Lv - \tau^* L\varphi_1 + Mw > \varepsilon v + \lambda_1(\Omega, q)w + Mw \geq 0 \quad \text{ in } \Omega.$$

From the strong maximum principle we have $w > 0$ in Ω or $w \equiv 0$ in $\mathbb{R}^N$. However, $w \equiv 0$ contradicts that $Lw > 0$. Now, from Hopf Lemma (Theorem 1.3), we know that $\partial_\nu v - \partial_\nu(\tau^*\varphi_1) = \partial_\nu w < 0$ on $\partial\Omega$. Thus, as $v \geq \tau^*\varphi_1 \geq 0$ and $\partial_\nu(\tau^*\varphi_1) > \partial_\nu v$ we can still find $\delta > 0$ such that $v \geq (\tau^* + \delta)\varphi_1 \geq 0$ in Ω. This contradicts the fact that τ^* is the largest possible in (5.14). Therefore (5.13) is false and we have

$$\lambda_1(\Omega, q) = \sup_{u \in \mathcal{V}(\Omega)} \inf_{x \in \Omega} \frac{Lu(x)}{u(x)}. \tag{5.15}$$

Furthermore, we know from part (a) of this theorem that

$$\varphi_1 \in \mathcal{V}(\Omega) \text{ and } L\varphi_1 = \lambda_1(\Omega, q)\varphi_1 \text{ in } \Omega.$$

Thus, the sup in (5.15) is indeed a max attained at φ_1. This completes the proof of **(e)** and hence the proof of Theorem 1.2. □

Appendix A. Proof of (3.13)

We sketch the proof of the fact (3.13) for the sake of completeness. Let $\psi \in \mathcal{C}^{1,\gamma}(\mathbb{R}^N)$ for some $\gamma < \alpha < 1$. We prove that $\psi \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ and

$$\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \le C\|\psi\|_{\mathcal{C}^{1,\gamma}(\mathbb{R}^N)}.$$

We make the following observations on the Hölder seminorm:

$$\begin{aligned}
[\psi]_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} &= \sup_{x,y\in\mathbb{R}^N, x\neq y} \frac{|\psi(x)-\psi(y)|}{|x-y|^\alpha} \\
&\le \sup_{x,y\in\mathbb{R}^N, 0<|x-y|\le 1} \frac{|\psi(x)-\psi(y)|}{|x-y|^\alpha} + 2\|\psi\|_{L^\infty(\mathbb{R}^N)} \\
&\le \sup_{x,y\in\mathbb{R}^N, 0<|x-y|\le 1} \frac{|\psi(x)-\psi(y)|}{|x-y|}|x-y|^{1-\alpha} + 2\|\psi\|_{L^\infty(\mathbb{R}^N)} \\
&\le \sup_{x,y\in\mathbb{R}^N, 0<|x-y|\le 1} \frac{|\psi(x)-\psi(y)|}{|x-y|} + 2\|\psi\|_{L^\infty(\mathbb{R}^N)} \\
&\le \|\nabla\psi\|_{L^\infty(\mathbb{R}^N)} + 2\|\psi\|_{L^\infty(\mathbb{R}^N)}.
\end{aligned}$$

Thus,

$$\|\psi\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^N)} \le 3\left(\|\nabla\psi\|_{L^\infty(\mathbb{R}^N)} + \|\psi\|_{L^\infty(\mathbb{R}^N)}\right) \le 3\|\psi\|_{\mathcal{C}^1(\mathbb{R}^N)} \le 3\|\psi\|_{\mathcal{C}^{1,\gamma}(\mathbb{R}^N)}$$

and this completes the proof of (3.13).

Data availability

No data was used for the research described in the article.

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