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Existence and regularity results for a Neumann problem with mixed local and nonlocal diffusion

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Abstract

In this paper, we consider an elliptic problem driven by a mixed local-nonlocal operator with drift and subject to nonlocal Neumann condition. We prove the existence and uniqueness of a solution $u \in W^{2,p}(\Omega)$ of the considered problem with L^p-source function when p and s are in a certain range.

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1. Introduction and main results

In this paper, we are interested in the study of the following problem

$$\begin{cases} Lu := -\Delta u(x) + (-\Delta)^s u(x) + q(x)\cdot\nabla u(x) + a(x)u = f(x) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \\ \mathcal{N}_s u = 0 & \text{on} \quad \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{1.1}$$

with a mixed diffusion and a new type of Neumann boundary conditions. The new boundary condition is $\mathcal{N}_s u = 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$, where $\mathcal{N}_s u$ — known as the nonlocal normal derivative of u is given by

$$\mathcal{N}_s u(x) = C_{N,s} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy \qquad x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{1.2}$$

The diffusion term is a superposition of the classical Laplacian (local diffusion) and the fractional Laplacian $(-\Delta)^s$ for certain values of $s \in (0, 1)$ that will be specified later. It is well known that the fractional Laplacian represents a nonlocal diffusion in the medium.

We recall that the operator $(-\Delta)^s$, with $s \in (0, 1)$, stands for the fractional Laplacian and it is defined for compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$ of class $\mathcal{C}^2$ by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy \tag{1.3}$$

with the same normalization constant $C_{N,s}$ as in (1.2) given by

$$C_{N,s} := \pi^{-\frac{N}{2}} 2^{2s} s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(1-s)}. \tag{1.4}$$

The boundary conditions in (1.1) consist of the classical Neumann boundary condition $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ (ν is the inward unit normal on $\partial\Omega$) and the nonlocal boundary condition $\mathcal{N}_s u = 0$ (see [8]) on $\mathbb{R}^N \setminus \Omega$. The classical Neumann condition states that there is no flux through the boundary of the domain. On the other hand, the nonlocal boundary condition $\mathcal{N}_s u = 0$ states that if a particle is in $\mathbb{R}^N \setminus \overline{\Omega}$, it may come back to any point $y \in \Omega$ with the probability density of jumping from x to y being proportional to $|x-y|^{-N-2s}$. A detailed description of (1.1) is given in [9]. The condition $\mathcal{N}_s u = 0$ is interpreted in [9] as a condition that arises from the superposition of Brownian and Lévy processes.

The PDE

$$-\Delta u(x) + (-\Delta)^s u + q(x)\cdot\nabla u(x) + a(x)u = f(x) \ \text{ in } \ \Omega \tag{1.5}$$

has been extensively studied when $q \equiv 0$ and the boundary condition is of Dirichlet type. That is, $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. Existence and regularity of solutions for (1.5), as well as maximum principles, are among the results obtained in [3], [4], [5], and [12], where the advection q is absent and the boundary condition is of Dirichlet type. The authors of this paper studied (1.5) in the recent work [6], where an advection term is present and (1.5) is coupled with the Dirichlet condition $u \equiv 0$ on $\mathbb{R} \setminus \overline{\Omega}$.

The recent work [7] considers (1.5) with $q \equiv 0$ and $a \equiv 0$ to provide spectral properties of the mixed diffusion operator. The work [8] considers a purely nonlocal diffusion and provides existence results for the problem with nonlocal Neumann conditions. It is important to note that [8] does not consider a PDE with a mixed diffusion and it does not account for advection.

The domain and the coefficients. Throughout this paper, we assume that the domain Ω is an open bounded connected subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The coefficients q and a are assumed to be uniformly Hölder continuous with $a \geq 0$ and not identically zero.

The normal derivative of u on $\partial\Omega$. Our solutions will, in general, be $C^1(\overline{\Omega})$ but the extension ($\widetilde{u}$ defined later) will not be sufficiently smooth. Hence to compute $\partial_\nu u(x)$ on $\partial\Omega$, we are using

$$\partial_\nu u(x) = \lim_{t \to 0^+} \frac{u(x_0 + t\nu(x_0)) - u(x_0)}{t},$$

where $\nu(x)$ is the unit inward normal to $\partial\Omega$ at $x \in \partial\Omega$.

We prove the following results for problem (1.1).

Theorem 1.1. *Let Ω be an open bounded set of $\mathbb{R}^N$ with smooth boundary and $f \in L^p(\Omega)$. Then,*

1. *if $\frac{N-1}{2N} < s < \frac{1}{2}$ and $N < p < \frac{1}{1-2s}$, problem (1.1) admits a unique solution $u \in W^{2,p}(\Omega)$;*
2. *if $p > N$ and $\frac{1}{2} \leq s < \frac{1}{2} + \frac{1}{2p}$, problem (1.1) admits a unique solution $u \in W^{2,p}(\Omega)$.*

2. The extension $\widetilde{u}$

We begin with the following extension definition for $u \in C^{0,1}(\overline{\Omega})$.

Definition 2.1. Let $u \in C^{0,1}(\overline{\Omega})$ and define the function $\widetilde{u}$ on $\mathbb{R}^N$ as

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \overline{\Omega} \\ u_1(x) & \text{if} \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{2.1}$$

where

$$u_1(x) := \frac{\displaystyle\int_\Omega \frac{u(y)}{|x-y|^{N+2s}}\,dy}{\displaystyle\int_\Omega \frac{1}{|x-y|^{N+2s}}\,dy}, \qquad x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{2.2}$$

Remark 2.2. We note that $\mathcal{N}_s\widetilde{u}(x) = 0$ for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$.

Let $K_\Omega : \Omega \times \Omega \to \mathbb{R}$ be the measurable (regional) kernel given by

$$K_\Omega(x, y) := \frac{1}{|x-y|^{N+2s}} + k_\Omega(x, y) \tag{2.3}$$

with

$$k_\Omega(x, y) := \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-z|^{N+2s}|y-z|^{N+2s} \int_\Omega \frac{1}{|z-z'|^{N+2s}}\, dz'}\, dz, \qquad x, y \in \Omega. \tag{2.4}$$

We now recall the following results that lead to integration by parts in a fractional setting from [2]:

Lemma 2.3 *([2]). Let $u, v : \mathbb{R}^N \to \mathbb{R}$ be two functions such that $\mathcal{N}_s v = 0$ on $\mathbb{R}^N \setminus \overline{\Omega}$. Then*

$$\begin{aligned} &\int_\Omega \int_\Omega (u(x) - u(y))(v(x) - v(y)) K_\Omega(x, y)\, dxdy \\ &\qquad = C_{N,s} \iint_{\mathcal{Q}} \frac{(\widetilde{u}(x) - \widetilde{u}(y))(v(x) - v(y))}{|x-y|^{N+2s}}\, dxdy \end{aligned} \tag{2.5}$$

From Lemma 2.3 and [8, Lemma 3.3], we deduce the integration by parts formula

$$\iint_{\mathcal{Q}} \frac{(\widetilde{u}(x) - \widetilde{u}(y))(v(x) - v(y))}{|x-y|^{N+2s}}\, dxdy = \int_\Omega v(-\Delta)^s \widetilde{u}\, dx + \int_{\mathbb{R}^N\setminus\Omega} v \mathcal{N}_s \widetilde{u}\, dx \tag{2.6}$$

for u and v being two C^2 bounded functions in $\mathbb{R}^N$.

Let $N < p < \infty$ and suppose $u \in W^{2,p}(\Omega)$ and hence u is $C^{0,1}(\overline{\Omega})$. Then note that $\widetilde{u}$ is smooth near x for any $x \notin \overline{\Omega}$. So the only real question on the smoothness of $\widetilde{u}$ is when $x \notin \Omega$ and $\delta(x) = dist(x, \partial\Omega) < 1$. For $x \notin \overline{\Omega}$, let $\hat{x} \in \partial\Omega$ be such that

$$|x - \hat{x}| = \inf_{z \in \partial\Omega} |z - x|.$$

Lemma 2.4. *Let Ω be an open bounded set of $\mathbb{R}^N$ with smooth boundary and let $N < p < \infty$ and suppose $u \in W^{2,p}(\Omega)$ with $\|u\|_{W^{2,p}} \le 1$. The following estimates are all independent of u.*

(i) *For $0 < s < 1/2$ there is some C such that for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\delta(x) < 1$ we have*

$$|\nabla u_1(x)| \le \frac{C}{(\delta(x))^{1-2s}}.$$

Therefore, $\widetilde{u} \in W^{1,q}_{loc}(\mathbb{R}^N)$ for all $1 < q < \dfrac{1}{1-2s}$ — after applying the co-area formula.

(ii) *If* $s = 1/2$ *then* $\widetilde{u} \in W^{1,q}_{loc}(\mathbb{R}^N)$ *for all* $1 \le q < \infty$.
(iii) *Let* $1/2 < s < 1$. *Then, there is some* $C > 0$ *such that* $|\nabla \widetilde{u}(x)| \le C$ *on* $\mathbb{R}^N$.

Proof. Let $x \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\delta(x) < 1$. For simplicity, we set

$$F(x) = \int_\Omega |x-y|^{-N-2s} dy.$$

Since $u \in W^{2,p}(\Omega)$ with $p > N$, we have that $u \in \mathcal{C}^{0,1}(\overline{\Omega})$. It follows from [8, Proposition 5.2] and the regularity of Ω that $\widetilde{u}$ is continuous in $\mathbb{R}^N$. Moreover, a direct computation shows that

$$\begin{aligned}\frac{(F(x))^2 \nabla u_1(x)}{N+2s} &= \int_\Omega (u(y)-u(\hat{x}))|x-y|^{-N-2s} dy \int_\Omega |x-y|^{-N-2s-2}(x-y)dy \\ &+ \int_\Omega |x-y|^{-N-2s} dy \int_\Omega (u(\hat{x})-u(y))|x-y|^{-N-2s-2}(x-y)dy.\end{aligned}$$

We now use the bound on u given by $|u(y)-u(\hat{x})| \le C_0|y-\hat{x}|$ to give

$$\begin{aligned}\frac{(F(x))^2 |\nabla u_1(x)|}{N+2s} &\le C_0 \int_\Omega \frac{|y-\hat{x}|}{|x-y|^{N+2s}} dy \int_\Omega \frac{1}{|x-y|^{N+2s+1}} dy \\ &+ C_0 \int_\Omega \frac{1}{|x-y|^{N+2s}} dy \int_\Omega \frac{|y-\hat{x}|}{|x-y|^{N+2s+1}} dy.\end{aligned}$$

Now note that for $y \in \Omega$ we have $|x-\hat{x}| \le |x-y|$ by the definition of $\hat{x}$ and hence we have

$$|y-\hat{x}| \le |y-x| + |x-\hat{x}| \le 2|y-x|.$$

It follows using the above inequality that

$$\begin{aligned}\frac{(F(x))^2 |\nabla u_1(x)|}{N+2s} &\le C_1 \int_\Omega \frac{1}{|x-y|^{N+2s-1}} dy \int_\Omega \frac{1}{|x-y|^{N+2s+1}} dy \\ &+ C_1 \int_\Omega \frac{1}{|x-y|^{N+2s}} dy \int_\Omega \frac{1}{|x-y|^{N+2s}} dy.\end{aligned}$$

Noticing that the second term in the right hand side is just $(F(x))^2$, we have

$$\frac{|\nabla u_1(x)|}{N+2s} \le C_1 + C_2 \frac{\int_\Omega \frac{1}{|x-y|^{N+2s-1}} dy \int_\Omega \frac{1}{|x-y|^{N+2s+1}} dy}{(F(x))^2}. \tag{2.7}$$

Now, it well known from [1, Lemma 2.1] that there are constants $C_1 > 0$ and $C_2 > 0$ such that for any $x \in \mathbb{R}^N \setminus \overline{\Omega}$, we have

$$C_1 \min\{(\delta(x))^{-2s}, (\delta(x))^{-N-2s}\} \le F(x) \le C_2 \min\{(\delta(x))^{-2s}, (\delta(x))^{-N-2s}\}. \tag{2.8}$$

With the assumption $\delta(x) < 1$, $x \in \mathbb{R}^N \setminus \overline{\Omega}$, this reduces to

$$C_1(\delta(x))^{-2s} \le F(x) \le C_2(\delta(x))^{-2s}. \tag{2.9}$$

We have the more general result that for $\tau > 0$ there is some $C_1, C_2 > 0$ such that for $x \notin \overline{\Omega}$ but with $\delta(x)$ small we have

$$\frac{C_1}{(\delta(x))^\tau} \le \int_\Omega \frac{1}{|x-y|^{N+\tau}} dy \le \frac{C_2}{(\delta(x))^\tau}. \tag{2.10}$$

We now distinguish three cases: $s \in (0, 1/2)$, $s = 1/2$ or $s \in (1/2, 1)$.

Case (i): $0 < s < 1/2$. Set $R_0 = 2\mathrm{diam}(\Omega)$, where $\mathrm{diam}(\Omega)$ is the diameter of Ω. Since Ω is bounded and $\delta(x) < 1$, $x \in \mathbb{R}^N \setminus \overline{\Omega}$, we have that $\Omega \subset B_{2R_0+1}(x)$. It follows that

$$\int_\Omega \frac{1}{|x-y|^{N+2s-1}} dy \le \int_{B_{2R_0+1}(0)} \frac{1}{|z|^{N+2s-1}} dz = C \int_0^{2R_0+1} \rho^{-2s}\, d\rho = C(2R_0+1)^{1-2s}. \tag{2.11}$$

Putting (2.9), (2.11) and (2.10) together, we get

$$\frac{|\nabla u_1(x)|}{N+2s} \le C_1 + \frac{C_3}{(\delta(x))^{1-2s}}.$$

We then apply the coarea formula now to get the desired result.

Case (ii): $s = 1/2$. We know from (2.7) that

$$|\nabla u_1(x)| \le C_1 + C_2 \frac{\int_\Omega \frac{1}{|x-y|^{N+2}} dy \int_\Omega \frac{1}{|x-y|^N} dy}{(F(x))^2},$$

and this gives combining (2.9) and (2.10),

$$|\nabla u_1(x)| \le C_1 + C_3 \int_\Omega \frac{1}{|x-y|^N} dy.$$

We now estimate the last term. Since $\Omega \subset B_{2R_0+1}(x)$ and $x \notin \Omega$ with $\delta(x) < 1$. Then we have

$$G(x) = \int_{y\in\Omega} \frac{1}{|x-y|^N} dy \le C \int_{\{z:\delta(x)\le|z|\le 2R_0+1\}} \frac{1}{|z|^N dz} = C \ln\left(\frac{2R_0+1}{\delta(x)}\right).$$

Then we have, after using the co-area formula,

$$\int_{\{x\notin\Omega,\delta(x)<1\}} G(x)^q dx \le C^q \int_{\{x\notin\Omega:\delta(x)<1\}} \left(\ln\left(\frac{2R_0+1}{\delta(x)}\right)\right)^q dx$$

$$= C^q \int_0^1 \left(\int_{\{x\notin\Omega:\delta(x)=t\}} \left\{\ln\left(\frac{2R_0+1}{\delta(x)}\right)\right\}^q d\sigma(x)\right) dt$$

$$= C^q \int_0^1 \left(\int_{\{x\notin\Omega:\delta(x)=t\}} \left\{\ln\left(\frac{2R_0+1}{t}\right)\right\}^q d\sigma(x)\right) dt$$

There is some $C>0$ such that $|\{x\notin\Omega:\delta(x)=t\}|\le C$ for all $0<t<1$, where $|A|$ refers the $N-1$ measure of A. From this we see (after doing a change of variables $r=1/t$ that we have

$$\int_{\{x\notin\Omega,\delta(x)<1\}} G(x)^q dx \le C\int_1^\infty \frac{(\ln((2R_0+1)r))^q}{r^2} dr,$$

and this is finite for any $1\le q<\infty$. Therefore, for any $x\in\mathbb{R}^N$ and $|x|\le R$, we have

$$\int_{B_R} |\nabla\widetilde{u}(x)|^q \, dx \le C_R$$

This shows that $\widetilde{u}\in W^{1,q}_{loc}(\mathbb{R}^N)$ for all $1<q<\infty$.

Case (iii): $1/2<s<1$**.** For $x\notin\overline{\Omega}$, we have from (2.7) that

$$\frac{|\nabla u_1(x)|}{N+2s} \le C_1 + C_2 \frac{\displaystyle\int_\Omega \frac{1}{|x-y|^{N+2s-1}}dy \int_\Omega \frac{1}{|x-y|^{N+2s+1}}dy}{(F(x))^2}$$

Now, since $\Omega\subset\mathbb{R}^N\setminus B_{\delta(x)}(x)$ for all $x\in\Omega^c$, we compute for $s>1/2$,

$$\int_\Omega \frac{1}{|x-y|^{N+2s-1}}\, dy \le \int_{\mathbb{R}^n\setminus B_{d(x)}} \frac{1}{|z|^{N+2s-1}}\, dz = C\int_{d(x)}^\infty \rho^{-2s}\, d\rho = C\delta(x)^{1-2s}.$$

This combined with (2.9) and (2.10) yield

$$\frac{|\nabla u_1(x)|}{N+2s} \le C_1 + C_4.$$

Hence, for all $x\in\mathbb{R}^N\setminus\overline{\Omega}$ we have

$$|\nabla u_1(x)| \le C. \quad \square$$

Lemma 2.5. *The following results hold:*

1. *Suppose* $\dfrac{N-1}{2N} < s < 1/2$ *and* $N < p < \dfrac{1}{1-2s}$. *Then, the mapping* $u \mapsto (-\Delta)^s \widetilde{u}$ *is continuous and compact from* $W^{2,p}(\Omega)$ *to* $L^p(\Omega)$.
2. *Suppose* $p > N$ *and* $1/2 \le s < \dfrac{1}{2} + \dfrac{1}{2p}$. *The mapping* $u \mapsto (-\Delta)^s \widetilde{u}$ *is continuous and compact from* $W^{2,p}(\Omega)$ *to* $L^p(\Omega)$.

Proof. 1. For the convenience of the reader we show the continuity and the compactness in separate steps. Let $u \in W^{2,p}(\Omega)$ with $\|u\|_{W^{2,p}(\Omega)} \le 1$ and let $x \in \Omega$ and then note we have

$$(-\Delta)^s \tilde{u}(x) = I + II + III$$

where

$$I(u)(x) = \int_{\{y\in\Omega:|y-x|\le 1\}} \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^{N+2s}} dy,$$

$$II(u)(x) = \int_{\{y\in\mathbb{R}^N\setminus\overline{\Omega}:|y-x|\le 1\}} \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^{N+2s}} dy,$$

$$\text{and } III(u)(x) = \int_{\{y:|y-x|\ge 1\}} \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^{N+2s}} dy.$$

Note that

$$|I(u)(x)| \le \int_{\{y\notin\Omega:|y-x|\le 1\}} \frac{C|x-y|}{|x-y|^{N+2s}} dy \le C_2,$$

since $s < \dfrac{1}{2}$. Also, note that

$$|III(u)(x)| \le \int_{\{y:|y-x|\ge 1\}} \frac{C}{|x-y|^{N+2s}} dy,$$

since $\widetilde{u}$ is bounded on $\mathbb{R}^N$ and hence $|III(u)(x)|$ is bounded in Ω by some C. We now estimate $II(u)$. Using $z = y - x$ we have

$$|II(u)(x)| \le \int_{|z|\le 1} \frac{|\widetilde{u}(x+z) - \widetilde{u}(x)|}{|z|^{N+2s}} dz.$$

Note that

$$|\widetilde{u}(x+z)-\widetilde{u}(x)| \le |z| \int_0^1 |\nabla \widetilde{u}(x+tz)|dt.$$

Hence, we have

$$\begin{aligned}
&\int_\Omega |II(u)(x)|^p dx \\
&\le \int_\Omega \left(\int_{|z|\le 1} \frac{1}{|z|^{N+2s-1}} \int_0^1 |\nabla \widetilde{u}(x+tz)|dtdz \right)^p dx \\
&\le C \int_\Omega \int_{|z|\le 1} \frac{1}{|z|^{N+2s-1}} \int_0^1 |\nabla \widetilde{u}(x+tz)|^p dtdzdx \quad \text{(Jensen's inequality applied twice)} \\
&= C \int_0^1 \int_{|z|\le 1} \frac{1}{|z|^{N+2s-1}} \left(\int_\Omega |\nabla \widetilde{u}(x+tz)|^p dx \right) dzdt \\
&\le C \int_0^1 \int_{|z|\le 1} \frac{1}{|z|^{N+2s-1}} \left(\int_{|x|\le R} |\nabla \widetilde{u}(x)|^p dx \right) dzdt,
\end{aligned}$$

for some large R and hence we have

$$\int_\Omega |II(u)(x)|^p dx \le C_2 \int_{|x|\le R} |\nabla \widetilde{u}(x)|^p dx.$$

Combining this with the results on $I(u)$ and $III(u)$, we see that $(-\Delta)^s \widetilde{u} \in L^p(\Omega)$ and is continuous from $W^{2,p}(\Omega)$.

We now consider the compactness. Since $W^{2,p}(\Omega)$ is a reflexive space its sufficient to show that if $u_m \rightharpoonup 0$ in $W^{2,p}(\Omega)$ then $(-\Delta)^s \widetilde{u}_m \to 0$ in $L^p(\Omega)$. Let $u_m \rightharpoonup 0$ in $W^{2,p}(\Omega)$ and hence it converges to zero in $W^{1,p}(\Omega)$ and uniformly in Ω.

$$\begin{aligned}
\frac{(F(x))^2 \nabla \widetilde{u}_m(x)}{N+2s} &= \int_\Omega (u_m(y)-u_m(\hat{x}))|x-y|^{-N-2s} dy \int_\Omega |x-y|^{-N-2s-2}(x-y)dy \\
&\quad + \int_\Omega |x-y|^{-N-2s} dy \int_\Omega (u_m(\hat{x})-u_m(y))|x-y|^{-N-2s-2}(x-y)dy.
\end{aligned}$$

From this we see that $|\nabla \widetilde{u}_m(x)| \to 0$ a.e. in $\mathbb{R}^N$ and note that we can use the result of Lemma 2.4 with the dominated convergence theorem to see that $\widetilde{u}_m \to 0$ in $W^{1,p}(B_R)$ for all $0 < R < \infty$.

Let $x \in \Omega$ and then note

$$(-\Delta)^s \widetilde{u}_m(x) = J_1(u_m)(x) + III(u_m)(x)$$

where

$$J_1(u_m)(x) = I(u_m)(x) + II(u_m)(x) = \int\limits_{\{y:|y-x|\leq 1\}} \frac{\widetilde{u}_m(x) - \widetilde{u}_m(y)}{|x-y|^{N+2s}} dy.$$

As before, we can write this as

$$\begin{aligned}
&\int\limits_{\Omega} |J_1(u_m)(x)|^p dx \\
&\leq C_1 \int\limits_0^1 \int\limits_{\{|z|\leq 1\}} \frac{1}{|z|^{N+2s-1}} \left(\int\limits_{x\in\Omega} |\nabla \widetilde{u}_m(x+tz)|^p dx \right) dz dx \\
&\leq C_2 \int\limits_{|x|<R} |\nabla \widetilde{u}_m(x)|^p dx,
\end{aligned}$$

for some large R and we know this goes to zero from the earlier results.

Let $R > 1$ be big and note we have

$$\begin{aligned}
|III(u_m)(x)| \leq& \int\limits_{\{y:1\leq|y-x|\leq R\}} \frac{|\widetilde{u}_m(x)| + |\widetilde{u}_m(y)|}{|x-y|^{N+2s}} dy \\
&+ \int\limits_{\{y:|y-x|\geq R\}} \frac{|\widetilde{u}_m(x)| + |\widetilde{u}_m(y)|}{|x-y|^{N+2s}} dy \\
\leq& \int\limits_{\{y:1\leq|y-x|\leq R\}} \frac{|\widetilde{u}_m(x)| + |\widetilde{u}_m(y)|}{|x-y|^{N+2s}} dy + CR^{-2s},
\end{aligned}$$

where C is from the fact that $|\widetilde{u}_m(x)| \leq C_1$ on $\mathbb{R}^N$ (independent of m). From this we see that

$$\begin{aligned}
\int\limits_{\Omega} |III(u_m)(x)|^p dx &\leq C_p R^{-2sp} + C_p \int\limits_{\Omega} \left(\int\limits_{1\leq|y-x|\leq R} \frac{|\widetilde{u}_m(x)| + |\widetilde{u}_m(y)|}{|x-y|^{N+2s}} dy \right)^p dx \\
&\leq C_p R^{-2sp} + C_p \int\limits_{\Omega} \left(2 \sup_{|\zeta|\leq R} |\widetilde{u}_m(\zeta)| \int\limits_{|z|\geq 1} \frac{1}{|z|^{N+2s}} dz \right)^p dx
\end{aligned}$$

and note that the second term goes to zero when $m \to \infty$. Hence, we have

$$\limsup_m \int_\Omega |III(u_m)(x)|^p dx \le C_p R^{-2sp},$$

and consequently $\int_\Omega |III(u_m)(x)|^p dx \to 0$ since we can set $R \to \infty$.

2. We now take $1/2 \le s < 1$ and for these cases we split the integral in the definition of the fractional Laplacian as

$$\begin{aligned}(-\Delta)^s \widetilde{u}(x) &= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{[\widetilde{u}(x) - \widetilde{u}(x+z)] + [\widetilde{u}(x) - \widetilde{u}(x-z)]}{|z|^{N+2s}} \, dz \\ &= \frac{C_{N,s}}{2} \sum_{i=1}^{4} \int_{A_x^i} \frac{[\widetilde{u}(x) - \widetilde{u}(x+z)] + [\widetilde{u}(x) - \widetilde{u}(x-z)]}{|z|^{N+2s}} \, dz \\ &\quad + \frac{C_{N,s}}{2} \int_{\{z \in \mathbb{R}^N : |z|>1\}} \frac{[\widetilde{u}(x) - \widetilde{u}(x+z)] + [\widetilde{u}(x) - \widetilde{u}(x-z)]}{|z|^{N+2s}} \, dz,\end{aligned}$$

where for $i = 1, \cdots, 4$, the sets A_x^i are defined as

$$\begin{aligned} A_x^1 &= \{z : |z| \le 1, x+z, x-z \in \Omega\}, \\ A_x^2 &= \{z : |z| \le 1, x+z \notin \Omega, x-z \in \Omega\}, \\ A_x^3 &= \{z : |z| \le 1, x+z \in \Omega, x-z \notin \Omega\}, \\ A_x^4 &= \{z : |z| \le 1, x+z \notin \Omega, x-z \notin \Omega\}. \end{aligned}$$

We first estimate the following

$$\int_{x\in\Omega} \left(\int_{\{z\in\mathbb{R}^N :\ |z|>1\}} \frac{|\widetilde{u}(x+z) + \widetilde{u}(x-z) - 2\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx.$$

First note that $\sup_{z\in\mathbb{R}^N} |\widetilde{u}(z)| \le \sup_{x\in\Omega} |u(x)|$ and hence the above quantity is bounded,

$$\int_{x\in\Omega} \left(\int_{\{z\in\mathbb{R}^N :\ |z|>1\}} \frac{|\widetilde{u}(x+z) + \widetilde{u}(x-z) - 2\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx \le C \|u\|^p_{W^{2,p}(\Omega)}.$$

We now estimate the integrals over A_x^i for $i = 1, \cdots, 4$.

The A_x^1 term. Let $1/2 \le s < 1$, $u \in W^{2,p}(\Omega)$ with $\|u\|_{W^{2,p}} \le 1$ and let v denote is $W^{2,p}(\Omega)$ extension to all of $\mathbb{R}^N$ which is compactly supported. Using a density argument we assume u and v are smooth. We want to estimate

$$\int_{x\in\Omega} \left(\int_{z\in A_x^1} \frac{|\widetilde{u}(x+z)+\widetilde{u}(x-z)-2\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx,$$

and note we can replace $\widetilde{u}$ with u in A_x^1 and then we can replace u with v. We now estimate this quantity. First note that for $|z| \le 1$ we have

$$|v(x+z)-v(x)-\nabla v(x)\cdot z| \le |z|^2 \int_0^1 \int_0^1 |D^2 v(x+t\tau z)| d\tau dt,$$

and from this we see that

$$|v(x+z)+v(x-z)-2v(x)| \le |z|^2 \int_0^1 \int_0^1 |D^2 v(x\pm t\tau z)| d\tau dt,$$

where the $\pm$ indicates there are two terms we need to consider. Then we have

$$\int_{x\in\Omega} \left(\int_{z\in A_x^1} \frac{|\widetilde{u}(x+z)+\widetilde{u}(x-z)-2\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx$$

is bounded above by

$$\int_{x\in\mathbb{R}^N} \left(\int_{z\in A_x^1} |z|^{-N-2s+2} \left(\int_0^1 \int_0^1 |D^2 v(x\pm t\tau z)| d\tau dt \right) dz \right)^p dx$$

and we can apply Jensen's inequality twice to get this bounded above by

$$C_1 \int_{x\in\mathbb{R}^N} \int_{z\in A_x^1} |z|^{-N-2s+2} \int_0^1 \int_0^1 |D^2 v(x\pm t\tau z)|^p d\tau dt dz dx$$

and by Fubini we see this bounded above by

$$C \int_0^1 \int_0^1 \int_{|z|\le 1} |z|^{-N-2s+2} \left(\int_{\mathbb{R}^N} |D^2 v(x\pm t\tau z)|^p dx \right) dz dt d\tau,$$

and the Extension Theorem (see [10, Theorem 1, Page 259]) we have the term in the brackets bounded by

$$\int_{\mathbb{R}^N} |D^2 v(x)|^p dx \leq C\|u\|^p_{W^{2,p}(\Omega)}$$

and this gives us the desired bound, that is

$$\int_{x\in\Omega} \left(\int_{z\in A^1_x} \frac{|\widetilde{u}(x+z)+\widetilde{u}(x-z)-2\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx \leq C\|u\|^p_{W^{2,p}(\Omega)}.$$

The A^i_x term for $i = 2, 3, 4$. Note that if $z \in A^i_x$ for $i = 2, 3, 4$, we must have $|z| > \delta(x)$. In what follows we will estimate

$$\int_{x\in\Omega} \left(\int_{z\in A^i_x} \frac{|\widetilde{u}(x+z)-\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx.$$

The same argument can be used to also estimate

$$\int_{x\in\Omega} \left(\int_{z\in A^i_x} \frac{|\widetilde{u}(x-z)-\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx$$

since the only fact we will use will be that $|z| > \delta(x)$. So to estimate the full quantity we group the three terms into the following pairings

$$[\widetilde{u}(x+z)-\widetilde{u}(x)] + [\widetilde{u}(x-z)-\widetilde{u}(x)]$$

and then estimate

$$\int_{x\in\Omega} \left(\int_{z\in A^i_x} \frac{|\widetilde{u}(x+z)+\widetilde{u}(x-z)-2\widetilde{u}(x)|}{|z|^{N+2s}} dz \right)^p dx \qquad \text{for} \quad i = 2, \cdots, 3.$$

We split the proof into two cases: $s = \frac{1}{2}$ and $s \in (1/2, 1)$.

The case $s = \frac{1}{2}$. Let $2 \leq i \leq 4$ and note that we have

$$\int_{x\in\Omega} \left(\int_{z\in A^i_x} \frac{|\widetilde{u}(x+z)-\widetilde{u}(x)|}{|z|^{N+1}} dz \right)^p dx$$

$$
\begin{aligned}
&\leq \int\limits_{x\in\Omega} \left(\int\limits_{z\in A_x^i} \frac{\int_0^1 |\nabla\widetilde{u}(x+tz)dt}{|z|^N} dz \right)^p dx \\
&= \int\limits_{x\in\Omega} \left(\int\limits_{z\in A_x^i} \frac{\int_0^1 |\nabla\widetilde{u}(x+tz)dt}{|z|^{N-\alpha}|z|^\alpha} dz \right)^p dx \quad \text{for some } \alpha>0 \text{ small, picked later} \\
&\leq \int\limits_{x\in\Omega} \frac{1}{(\delta(x))^{\alpha p}} \left(\int\limits_{z\in A_x^i} \frac{\int_0^1 |\nabla\widetilde{u}(x+tz)dt}{|z|^{N-\alpha}} dz \right)^p dx \\
&\leq C \int\limits_{x\in\Omega} \frac{1}{(\delta(x))^{\alpha p}} \left(\int\limits_{z\in A_x^i} \frac{\int_0^1 |\nabla\widetilde{u}(x+tz)|^p dt}{|z|^{N-\alpha}} dz \right) dx \quad \text{(applying Jensen's inequality twice)} \\
&\leq C \int\limits_{x\in\Omega} \frac{1}{(\delta(x))^{\alpha p}} \left(\int\limits_{\delta(x)<|z|\leq 1} \frac{\int_0^1 |\nabla\widetilde{u}(x+tz)|^p dt}{|z|^{N-\alpha}} dz \right) dx \\
&= C \int\limits_0^1 \int\limits_{|z|\leq 1} \frac{1}{|z|^{N-\alpha}} \left(\int\limits_{\{x\in\Omega:\delta(x)\leq|z|\}} \frac{|\nabla\widetilde{u}(x+tz)|^p}{(\delta(x))^{\alpha p}} dx \right) dz dt.
\end{aligned}
$$

We now fix $0<|z|\leq 1$ and $0<t<1$ and note for $1<q<\infty$ we have

$$
\int\limits_{\{x\in\Omega:\delta(x)\leq|z|\}} \frac{|\nabla\widetilde{u}(x+tz)|^p}{(\delta(x))^{\alpha p}} dx \leq \left(\int\limits_\Omega |\nabla\widetilde{u}(x+tz)|^{pq} dx \right)^{\frac{1}{q}} \left(\int\limits_\Omega \frac{1}{(\delta(x))^{\alpha p q'}} dx \right)^{\frac{1}{q'}}
$$

and so for fixed q we can take $\alpha>0$ small enough so that $\alpha p q' < 1$ and the internal involving the distance function is bounded and the other integral is bounded (independent of z and t) after considering the earlier gradient bound on $\widetilde{u}$. This shows that

$$
\int\limits_{x\in\Omega} \left(\int\limits_{z\in A_x^i} \frac{|\widetilde{u}(x+z)-\widetilde{u}(x)|}{|z|^{N+1}} dz \right)^p dx
$$

is bounded.

The case $s\in(1/2,1)$**.** Let $2\leq i\leq 4$ and u be as above. Recalling for $z\in A_x^i$ we have $|z|>\delta(x)$ we have

$$
\begin{aligned}
&\int_{x\in\Omega}\left(\int_{z\in A_x^i}\frac{|\widetilde{u}(x+z)-\widetilde{u}(x)|}{|z|^{N+2s}}dz\right)^p dz\\
&\leq\int_{\Omega}\left(\int_{A_x^i}\frac{\int_0^1|\nabla\widetilde{u}(x+tz)|dt}{|z|^{N+2s-1}}dz\right)^p dx\\
&=\int_{\Omega}\left(\int_{A_x^i}\frac{\int_0^1|\nabla\widetilde{u}(x+tz)|dt}{|z|^{N+2s-1-\alpha}|z|^{\alpha}}dz\right)^p dx \qquad \alpha>0\\
&\leq\int_{\Omega}\frac{1}{(\delta(x))^{\alpha p}}\left(\int_{A_x^i}\frac{\int_0^1|\nabla\widetilde{u}(x+tz)|dt}{|z|^{N+2s-1-\alpha}}dz\right)^p dx\\
&\leq C\int_{\Omega}\frac{1}{(\delta(x))^{\alpha p}}\left(\int_{A_x^i}\frac{\int_0^1|\nabla\widetilde{u}(x+tz)|^p dt}{|z|^{N+2s-1-\alpha}}dz\right)dx \qquad \text{(Jensen's inequality applied twice)}
\end{aligned}
$$

If we now assume that $|\nabla\widetilde{u}|\leq C$ then we get this is bounded above by

$$
C\int_{x\in\Omega}\frac{1}{\delta(x)^{\alpha p}}\left(\int_{z\in A_x^i}\frac{1}{|z|^{N+2s-1-\alpha}}dz\right)dx,
$$

and since $A_x^i\subset\{z:|z|\leq 1,|z|>\delta(x)\}$ then to have this bounded its sufficient that $\alpha p<1$ and $2s-1-\alpha<0$. Hence we see its sufficient that $2s-1<\frac{1}{p}$. The compactness proof follows the same ideas as the previous range of s. □

The following result is a maximum principle that we will use in the proof of existence of a solution.

Theorem 2.6. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, q and a be uniformly Hölder continuous with $a(x)\geq 0$ in Ω. Suppose $u\in W^{2,p}(\Omega)$ is a solution of*

$$
\begin{cases}
Lu=0 & in \quad \Omega,\\
\dfrac{\partial u}{\partial\nu}=0 & on \quad \partial\Omega,\\
\mathcal{N}_s u=0 & on \quad \mathbb{R}^N\setminus\overline{\Omega}.
\end{cases}
\tag{2.12}
$$

Then, $\widetilde{u}\equiv 0$ in $\mathbb{R}^N$.

Proof. Let u denote the solution and, for notation, we also let u denote $\widetilde{u}$ outside of Ω. Suppose $x_0 \in \overline{\Omega}$ such that $u(x_0) = \inf_{\Omega} u$. We first rule out $x_0 \in \partial\Omega$. If $x_0 \in \partial\Omega$, then we take $x_m \notin \overline{\Omega}$ such that $x_m \to x_0$ as $m \to +\infty$. Using the nonlocal boundary condition we get

$$0 = \int_{\Omega} \frac{u(y) - u(x_m)}{|y - x_m|^{N+2s}} dy.$$

Passing to the limit, we see that

$$0 = \int_{\Omega} \frac{u(y) - u(x_0)}{|y - x_0|^{N+2s}} dy,$$

which shows that $u = C = const.$ is constant in Ω and hence constant in $\mathbb{R}^N$. Then note from the equation we have $a(x)C = 0$ in Ω and this shows that $u = C = 0$ provided $a(x)$ is not identically zero which we have assumed and hence we are done.

We now suppose $x_0 \in \Omega$ is such that $u(x_0) = \inf_{\Omega} u$ and we suppose $u(x_0) < 0$.

We can also suppose that $u(x_0) < \inf_{\partial\Omega} u$. Note that, from the definition of u_1, we get $u(x) > u(x_0)$ for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$.

Fix $\sigma > 0$ such that $u(x_0) + 10\sigma < \min_{\partial\Omega} u$ and we also assume $u(x_0) + 10\sigma < 0$. For $\varepsilon > 0$ small set $\Omega_\varepsilon = \{x \in \Omega : \delta(x) > \varepsilon\}$ and $\Gamma_\varepsilon = \{x \in \Omega : \delta(x) < \varepsilon\}$. By continuity there is some $\varepsilon_0 > 0$ such that

$$u(x_0) + 8\sigma < \inf_{\Gamma_{\varepsilon_0}} u.$$

For $x \notin \overline{\Omega}$ we have

$$\begin{aligned}
u(x) \int_{\Omega} \frac{1}{|x-y|^{N+2s}} dy &= \int_{\Gamma_{\varepsilon_0}} \frac{u(y)}{|x-y|^{N+2s}} dy + \int_{\Omega \setminus \Gamma_{\varepsilon_0}} \frac{u(y)}{|x-y|^{N+2s}} dy \\
&\geq (u(x_0) + 8\sigma) \int_{\Gamma_{\varepsilon_0}} \frac{1}{|x-y|^{N+2s}} dy + u(x_0) \int_{\Omega \setminus \Gamma_{\varepsilon_0}} \frac{1}{|x-y|^{N+2s}} dy \\
&= u(x_0) \int_{\Omega} \frac{1}{|x-y|^{N+2s}} dy + 8\sigma \int_{\Gamma_{\varepsilon_0}} \frac{1}{|x-y|^{N+2s}} dy
\end{aligned}$$

which gives

$$u(x) \geq u(x_0) + \frac{8\sigma \int_{\Gamma_{\varepsilon_0}} \frac{1}{|x-y|^{N+2s}} dy}{\int_{\Omega} \frac{1}{|x-y|^{N+2s}} dy}.$$

From this we can show there is some $c_{\varepsilon_0} > 0$ (without loss of generality we can take $c_{\varepsilon_0} < 1$) such that $u(x) \geq u(x_0) + 8\sigma c_{\varepsilon_0}$ for all $x \notin \overline{\Omega}$.

Let η denote a standard radial mollifier with η_ε the appropriately scaled function whose support is $\overline{B_\varepsilon}$ and set $u^\varepsilon(x) = (\eta_\varepsilon * u)(x)$ for $x \in \mathbb{R}^N$. For all $0 < \varepsilon < \varepsilon_0$ we have

$$\inf_{\partial\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}} \geq \inf_{\Gamma_\varepsilon} u,$$

and also we have $\inf_{\Omega_\tau} u^\tau \to u(x_0)$ as $\tau \to 0$. From this we see there is some $0 < \varepsilon_1 < \varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_1$ we have

$$\inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}} + 6\sigma \leq u(x_0) + 8\sigma < \inf_{\Gamma_{\varepsilon_0}} u,$$

but by the monotonicity of $\varepsilon \mapsto \inf_{\Gamma_\varepsilon} u$ we have

$$\inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}} + 6\sigma < \inf_{\Gamma_{\varepsilon_0}} u \leq \inf_{\Gamma_\varepsilon} u$$

for all $0 < \varepsilon < \varepsilon_1$ and hence we have

$$\inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}} + 6\sigma < \inf_{\partial\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}},$$

for $0 < \varepsilon < \varepsilon_1$ and hence the minimum is contained in the interior of $\Omega_{\frac{\varepsilon}{2}}$. We now want to show that $\min_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}} \leq u^{\frac{\varepsilon}{2}}(x)$ for all $x \in \mathbb{R}^N$. Let $0 < \varepsilon < \frac{\varepsilon_1}{10}$ with:

$$u^{\frac{\varepsilon}{2}}(x_0) < u(x_0) + 8\sigma c_{\varepsilon_0}. \tag{2.13}$$

We consider the three cases:

(i) $x \in \Omega$ with $\delta(x) < \frac{\varepsilon}{2}$, (ii) $x \notin \Omega$ with $\delta(x) < \frac{\varepsilon}{2}$ and (iii) $x \notin \Omega$ with $\delta(x) > \frac{\varepsilon}{2}$.

Case (i). Here, we have $x \in \Omega$ with $\delta(x) < \frac{\varepsilon}{2}$. So in this case we have

$$u^{\frac{\varepsilon}{2}}(x) = \int_{|y-x|<\frac{\varepsilon}{2}} \eta_{\frac{\varepsilon}{2}}(y-x)u(y)dy,$$

and note the integral can be decomposed as

$$\int_{|y-x|<\frac{\varepsilon}{2}, y\in\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)u(y)dy + \int_{|y-x|<\frac{\varepsilon}{2}, y\notin\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)u(y)dy$$

and from this we see that

$$
\begin{aligned}
u^{\frac{\varepsilon}{2}}(x) &\geq (u(x_0) + 8\sigma c_{\varepsilon_0}) \int_{|y-x|<\frac{\varepsilon}{2}} \eta_{\frac{\varepsilon}{2}}(y-x)dy \\
&= u(x_0) + 8\sigma c_{\varepsilon_0} \\
&> u^{\frac{\varepsilon}{2}}(x_0) \\
&\geq \inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}}.
\end{aligned}
$$

Case (ii). We have

$$
\begin{aligned}
(\tilde{u})^{\frac{\varepsilon}{2}}(x) = &\int_{|y-x|<\frac{\varepsilon}{2}, y\in\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)\tilde{u}(y)dy + \int_{|y-x|<\frac{\varepsilon}{2}, y\notin\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)\tilde{u}(y)dy \\
&\geq \inf_{\Gamma_{\frac{\varepsilon}{2}}} u \int_{|y-x|<\frac{\varepsilon}{2}, y\in\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)dy + (u(x_0) + 8\sigma c_{\varepsilon_0}) \int_{|y-x|<\frac{\varepsilon}{2}, y\notin\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)dy \\
&\geq \left(\inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}} + 6\sigma\right) \int_{|y-x|<\frac{\varepsilon}{2}, y\in\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)dy + (u(x_0) + 8\sigma c_{\varepsilon_0}) \int_{|y-x|<\frac{\varepsilon}{2}, y\notin\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)dy \\
&\geq \left(\inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}} + 6\sigma\right) \int_{|y-x|<\frac{\varepsilon}{2}, y\in\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)dy + u^{\frac{\varepsilon}{2}}(x_0) \int_{|y-x|<\frac{\varepsilon}{2}, y\notin\Omega} \eta_{\frac{\varepsilon}{2}}(y-x)dy \\
&\geq \inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}}
\end{aligned}
$$

because $u^{\frac{\varepsilon}{2}}(x_0) \geq \inf_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}}$ and $\int_{|y-x|<\frac{\varepsilon}{2}} \eta_{\frac{\varepsilon}{2}}(y-x)dy = 1.$

Case (iii). This follows similarly.

From the above we have, for small enough ε, that $u^{\frac{\varepsilon}{2}}(x_{\frac{\varepsilon}{2}}) = \min_{\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}}$ (some $x_{\frac{\varepsilon}{2}} \in \Omega_{\frac{\varepsilon}{2}}$) and $u^{\frac{\varepsilon}{2}}(x_{\frac{\varepsilon}{2}}) + 6\sigma < \inf_{\partial\Omega_{\frac{\varepsilon}{2}}} u^{\frac{\varepsilon}{2}}$. Also note we have (take $\tau = \varepsilon/2$) $L(u^\tau)(x) = 0$ in Ω_τ and at x_τ we have

$$
-\Delta u^\tau(x_\tau) + (-\Delta)^s u^\tau(x_\tau) + q(x_\tau) \cdot \nabla u^\tau(x_\tau) = -a(x_\tau)u^\tau(x_\tau) \geq 0. \tag{2.14}
$$

But $-\Delta u^\tau(x_\tau) \leq 0$, $\nabla u^\tau(x_\tau) = 0$ and note that

$$
(-\Delta)^s u^{\frac{\varepsilon}{2}}(x_{\frac{\varepsilon}{2}}) = \int_{y\in\mathbb{R}^N} \frac{u^{\frac{\varepsilon}{2}}(x_{\frac{\varepsilon}{2}}) - u^{\frac{\varepsilon}{2}}(y)}{|x_{\frac{\varepsilon}{2}} - y|^{N+2s}} dy,
$$

and note $y \mapsto u^{\frac{\varepsilon}{2}}(x_{\frac{\varepsilon}{2}}) - u^{\frac{\varepsilon}{2}}(y)$ is continuous in y on $\mathbb{R}^N$, nonpositive and not identically zero. From this we see that $(-\Delta)^s u^{\frac{\varepsilon}{2}}(x_{\frac{\varepsilon}{2}}) < 0$ and this contradicts (2.14). The proof of Theorem 2.6 is completed. □

3. Proof of existence of a solution

This section is dedicated to the proof of Theorem 1.1. We apply the method of continuity under the Neumann boundary condition.

Proof of Theorem 1.1. For $u \in W^{2,p}(\Omega)$ and $\gamma \in \mathbb{R}$, we define

$$L_\gamma u(x) = -\Delta u + \gamma(-\Delta)^s \tilde{u}(x) + a(x)u(x) + q \cdot \nabla u(x), \quad x \in \Omega$$

and we consider the family of indexed problems

$$\begin{cases} L_\gamma u = f & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

Let $\mathcal{A}$ be the set

$$\mathcal{A} := \left\{ \begin{matrix} \gamma \in [0,1] : \exists C_\gamma > 0 \text{ such that for all } f \in L^p(\Omega), (3.1) \text{ has a solution} \\ u \in W^{2,p}(\Omega) \text{ such that } \|u\|_{W^{2,p}(\Omega)} \leq C_\gamma \|f\|_{L^p(\Omega)} \end{matrix} \right\}. \tag{3.2}$$

In (3.2), we take the constant C_γ to be the smallest constant such that $\|u\|_{W^{2,p}(\Omega)} \leq C_\gamma \|f\|_{L^p(\Omega)}$ holds for all functions $f \in L^p(\Omega)$. In other words, if $C_\gamma > \varepsilon > 0$ then there exists $f_\varepsilon \in C^{0,\alpha}(\overline{\Omega})$ such that

$$\|u\|_{W^{2,p}(\Omega)} \geq (C_\gamma - \varepsilon)\|f_\varepsilon\|_{L^p(\Omega)}. \tag{3.3}$$

By classical theory, $0 \in \mathcal{A}$ (see [11, section 2.4]). Our goal is to show that $\mathcal{A}$ is both open and closed and since $[0,1]$ is connected we then see that $\mathcal{A} \in \{\emptyset, [0,1]\}$, and since its non empty, we must have $\mathcal{A} = [0,1]$. In particular $1 \in \mathcal{A}$ which corresponds to the result we are trying to prove.

$\mathcal{A}$ is closed. Let $\gamma_m \in \mathcal{A}$ with $\gamma_m \to \gamma$ and let $C_m = C_{\gamma_m}$ denote constant associated with C_m. We first consider the case where $\{C_m\}$ is bounded. Let $f \in L^p(\Omega)$ with $\|f\|_{L^p} = 1$. Since $\gamma_m \in \mathcal{A}$ there is some $u_m \in W^{2,p}(\Omega)$ which satisfies (3.1), with γ_m in place of γ, and $\|u_m\|_{W^{2,p}} \leq C_m\|f\|_{L^p} = C_m$. Note we can rewrite the problem as

$$-\Delta u_m + a u_m + q \cdot \nabla u_m = f - \gamma_m(-\Delta)^s \widetilde{u}_m \quad \text{in } \Omega, \tag{3.4}$$

with $\partial_\nu u_m = 0$ on $\partial\Omega$. Since $\{C_m\}$ is bounded then we have $\{u_m\}$ bounded in $W^{2,p}(\Omega)$ and by passing to a subsequence we can assume that $u_m \rightharpoonup u$ in $W^{2,p}(\Omega)$ and $\|u\|_{W^{2,p}} \leq \liminf_m \|u_m\|_{W^{2,p}} \leq C_1$. Also note by our earlier compactness result we have $(-\Delta)^s \widetilde{u}_m \to (-\Delta)^s \widetilde{u}$ in $L^p(\Omega)$ and this (along with the above weak $W^{2,p}$ convergence) is sufficient convergence to pass to the limit in (3.4). This shows that $\gamma \in \mathcal{A}$.

We now consider the case of $C_m \to \infty$. Then, there is some $f_m \in L^p(\Omega)$ and $u_m \in W^{2,p}(\Omega)$ which solves $L_{\gamma_m} u_m = f_m$ in Ω with $\partial_\nu u_m = 0$ on $\partial\Omega$ and

$$\|u_m\|_{W^{2,p}} \geq (C_m - 1)\|f_m\|_{L^p}.$$

By normalizing we can assume $\|u_m\|_{W^{2,p}(\Omega)} = 1$ and hence $\|f_m\|_{L^p} \to 0$. By passing to subsequences we can assume that $u_m \rightharpoonup u$ in $W^{2,p}(\Omega)$ and strongly in $W^{1,p}(\Omega)$. As before we rewrite the equation for u_m by

$$-\Delta u_m + a u_m + q \cdot \nabla u_m = f_m - \gamma_m (-\Delta)^s \widetilde{u}_m \quad \text{in } \Omega, \tag{3.5}$$

with $\partial_\nu u_m = 0$ on $\partial\Omega$. If $u = 0$ then note the right hand side of (3.5) converges to zero in $L^p(\Omega)$ and by standard elliptic theory we have $u_m \to 0$ in $W^{2,p}(\Omega)$ which contradicts the normalization of u_m. We now assume $u \neq 0$. By compactness we can pass to the limit in (3.5) to see that $u \in W^{2,p}(\Omega)\backslash\{0\}$ satisfies $L_\gamma u = 0$ in Ω with $\partial_\nu u = 0$ on $\partial\Omega$ which contradicts Theorem 2.6.

$\mathcal{A}$ **is open.** Let $\gamma_0 \in \mathcal{A}$ and take $|\varepsilon|$ small; when $\gamma_0 \in \{0, 1\}$ we need to restrict the sign of ε. Our goal is to show that $\gamma = \gamma_0 + \varepsilon \in \mathcal{A}$. Fix $f \in L^p(\Omega)$ with $\|f\|_{L^p} = 1$ and since $\gamma_0 \in \mathcal{A}$ there is some $v_0 \in W^{2,p}(\Omega)$ which solves $L_{\gamma_0} v_0 = f$ in Ω with $\partial_\nu v_0 = 0$ on $\partial\Omega$. We look for a solution of (3.1) of the form $u = v_0 + \phi$. Writing out the details one sees we need $\phi \in W^{2,p}(\Omega)$ to satisfy

$$L_{\gamma_0}\phi = -\varepsilon(-\Delta)^s \widetilde{v}_0 - \varepsilon(-\Delta)^s \widetilde{\phi} \quad \text{in } \Omega, \tag{3.6}$$

with $\partial_\nu \phi = 0$ on $\partial\Omega$. Define the operator $J_\varepsilon(\phi) = \psi$ where ψ satisfies

$$L_{\gamma_0}\psi = -\varepsilon(-\Delta)^s \widetilde{v}_0 - \varepsilon(-\Delta)^s \widetilde{\phi} \quad \text{in } \Omega, \tag{3.7}$$

with $\partial_\nu \psi = 0$ on $\partial\Omega$. We claim that for small enough ε that J_ε is a contraction mapping on $W^{2,p}(\Omega)$ and hence by the Contraction Mapping Principle there is some $\phi \in W^{2,p}(\Omega)$ with $J_\varepsilon(\phi) = \phi$. From (3.6) one would then get a $W^{2,p}(\Omega)$ bound on ϕ and hence we'd get the desired bound on u. We first note that J_ε is into $W^{2,p}(\Omega)$ after noting the right hand side of (3.6) belongs to $L^p(\Omega)$. Let $\phi_i \in W^{2,p}(\Omega)$ and $\psi_i = J_\varepsilon(\phi_i)$ and then note we have

$$L_{\gamma_0}(\psi_2 - \psi_1) = -\varepsilon(-\Delta)^s(\widetilde{\phi}_2 - \widetilde{\phi}_1) \quad \text{in } \Omega,$$

with the desired boundary condition. Then we have

$$\|\psi_2 - \psi_1\|_{W^{2,p}} \leq C_{\gamma_0}|\varepsilon| \|(-\Delta)^s(\widetilde{\phi}_2 - \widetilde{\phi}_1)\|_{L^p} \leq C_{\gamma_0}|\varepsilon| C \|\phi_2 - \phi_1\|_{W^{2,p}},$$

and hence we see of $|\varepsilon|$ small that J_ε is a contraction on $W^{2,p}(\Omega)$ and this completes the proof of Theorem 1.1. □

Data availability

No data was used for the research described in the article.

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