Properties of the Gamma Function and Riemann Zeta Function

by

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Abstract

The two special functions known as the gamma function and the Riemann zeta function are both functions initially defined through a relatively simple rule (the gamma function defined as an integral and the Riemann zeta function as a sum) on a half-plane of the complex plane. Then, analytic continuation is used in order to derive the rest of the functions. This paper explores how these analytic continuations come about, as well as various interesting properties and values of the complete gamma and Riemann zeta functions.

Table of Contents

Foreword

This article assumes mathematical knowledge at roughly a third-year undergraduate level. In particular, the use of calculus and complex analysis will be extensive. It would be quite helpful to have read the majority of [6] or an equivalent textbook (in particular, subjects such as differential and integral calculus as well as infinite sums should be understood well), as well as up to chapter three of [3] (in particular, subjects such as complex integration and what it means for a function to be analytic or meromorphic will be used). A corollary of various integration tests from [6] that will be used but may not be immediately obvious will be stated and proved here.

Lemma 0.1. *Let $I \in \mathbb{R}$ be an interval. As well, let $f : I \to \mathbb{R}$. Suppose*

$$\int_I (f(x))\,dx \text{ absolutely converges.}$$

Suppose that $g : I \to \mathbb{R}$ and that there exists $M > 0$ such that for all $x \in I$,

$|g(x)| \leq M$. *Then,*

$$\int_I (f(x)g(x))\,dx \text{ \textit{converges.}}$$

Proof. It should be noted that if

$$\int_I (f(x))\,dx \text{ abslutely converges, then}$$
$$M\int_I (f(x))\,dx \text{ absolutely converges.}$$

Thus, as a constant outside of an integral may be made part of an integrand,

$$M\int_I |f(x)|\,dx = \int_I |Mf|\,dx > \left|\int_I (f(x)g(x))\,dx\right|.$$

Thus,

$$\int_I (f(x)g(x))\,dx \text{ converges.}$$

□

Fourier series will be used, so one should have at least cursory knowledge of what a Fourier series is. They are briefly mentioned in textbooks such as [7].

As well, there is a bit of custom notation that will be used in this article. First, here are the notations used for various integer subsets, and then an ease-of-writing notation will be introduced:

$$\begin{aligned}
\mathbb{Z}^+ &= \{1, 2, 3, \dots\} = \{n \in \mathbb{Z} \mid n > 0\};\\
\mathbb{Z}^- &= \{-1, -2, -3, \dots\} = \{n \in \mathbb{Z} \mid n < 0\};\\
\mathbb{N} &= \{0, 1, 2, \dots\} = \{n \in \mathbb{Z} \mid n \geq 0\};\\
\mathbb{N}^- &= \{0, -1, -2, \dots\} = \{n \in \mathbb{Z} \mid n \leq 0\}.
\end{aligned}$$

Notation 0.1. Let $\Lambda \subseteq \mathbb{R}$. Denote the portion of the complex plane such that all numbers therein have real part within Λ by

$$\mathcal{R}\Lambda = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \in \Lambda\}. \tag{1}$$

For example, $\mathcal{R}[0, 4) = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \in [0, 4)\}$.

This article is meant to be a compendium of sorts, cataloguing various properties of the two special functions named in the title as well as how the two seemingly unrelated functions are intertwined.

Chapter 1

The Gamma Function

1.1 Analytic Continuation

There exist many complex functions whose domains are, in a sense, not as large as they could be. For an example, let the set $S = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc in the complex plane, and let $f : S \to \mathbb{C}$ be defined by

$$f(z) = \frac{z}{2}.$$

Obviously, there exist $z \in \mathbb{C}$ such that $z \notin S$ (as an example, $z = 5$), so the domain of f could potentially be "extended" to include more values.

There are an uncountably infinite number of ways to extend this function to a larger domain. Two such ways shall be given in the form of $g : \mathbb{C} \to \mathbb{C}$ and $h : \mathbb{C} \to \mathbb{C}$, defined by

$$g(z) = \frac{z}{2};$$
$$h(z) = \begin{cases} \dfrac{z}{2} & , \ |z| < 1 \\ z + 1 & , \ |z| \geq 1. \end{cases}$$

When $|z| < 1$, $g(z) = h(z) = f(z)$, so both g and h "extend" f to a larger domain (said larger domain being all of $\mathbb{C}$); however, g seems to be a more "natural" fit. For this example, it can be chalked down to g simply being defined by the same elementary function, but it would be useful to have a way to determine if one method of extending a function is "better" than another using precise language.

One thing that differentiates g from h is that g is an analytic function, whereas h is not, as $h(z)$ is discontinuous wherever $|z| = 1$. This property thus motivates the study of what is known as analytic continuation.

Definition 1.1 (Analytic Continuation). Let $U_f \subseteq \mathbb{C}$ and $U_g \subseteq \mathbb{C}$ be open and connected, with $U_f \cap U_g \neq \emptyset$. Let $f : U_f \to \mathbb{C}$ and $g : U_g \to \mathbb{C}$ be analytic. Then, g is an **analytic continuation** of f on U_g if for all $z \in U_f \cap U_g$, $g(z) = f(z)$. [4]

This definition of analytic continuation is in a sense too broad, as the only analytic continuation that will be done in this article will be from a set to a superset of said set, but it is useful to use already-established conditions.

Analytic continuation provides a “natural” method of extending functions beyond their original domains. In the above example, g is an analytic continuation of f on $\mathbb{C}$, whereas h is not. Analytic continuation is used in order to define both the gamma function and the Riemann zeta function.

1.2 Deriving the Gamma Function

The gamma function is a function defined on the open half-plane with positive real part (i.e., $\mathcal{R}(0,\infty)$) by an integral expresion and everywhere else (save the nonpositive integers) through analytic continuation. Before studying the function in detail, it is useful to note a few things about a particular integral.

Lemma 1.1. *For all* $z \in \mathcal{R}(0,\infty)$,

$$\int_0^\infty \left(e^{(-t)}t^{(z-1)}\right) dt \textit{ converges.}$$

Proof. Suppose $z > 0$. Then, the integral can be decomposed as follows:

$$\int_0^\infty \left(e^{(-t)}t^{(z-1)}\right) dt = \int_0^1 \left(e^{(-t)}t^{(z-1)}\right) dt + \int_1^\infty \left(e^{(-t)}t^{(z-1)}\right) dt. \tag{1.1}$$

The exponential function has more effect on convergence than any power function, so the second term in the righthand side of (1.1) converges, and for all $t \in (0,1)$, $\left|e^{(-t)}\right| < 1$, so, by Lemma 0.1,

$$\int_0^\infty \left(e^{(-t)}t^{(z-1)}\right) dt \text{ coverges if } \int_0^1 \left(t^{(z-1)}\right) dt \text{ converges,}$$

which it does.

Suppose more generally that $z \in \mathcal{R}(0,\infty)$. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$, where $z = x + iy$. Then, the integral becomes

$$\begin{aligned}\int_0^\infty \left(e^{(-t)}t^{(x+iy-1)}\right) dt &= \int_0^\infty \left(e^{(-t)}t^{(x-1)}t^{iy}\right) dt \\ &= \int_0^\infty \left(e^{(-t)}t^{(x-1)}e^{(iy\ln(|t|))}\right) dt \\ &= \int_0^\infty \left(e^{(-t)}t^{(x-1)}\left(e^{(iy)}\right)^{(\ln(|t|))}\right) dt.\end{aligned}$$

Examining the absolute value of the integral, and noting that, for any positive t, $\left|\left(e^{(iy)}\right)\right|^{(\ln(|t|))} = 1^{(\ln(|t|))} = 1$, and that t, $e^{(-t)}$, and $t^{(x-1)}$

are all positive quantites, leads to the following result:

$$\left| \int_0^\infty \left(e^{(-t)} t^{(x+iy-1)} \right) dt \right| \leq \int_0^\infty \left(\left| e^{(-t)} t^{(x-1)} \right| \right) dt = \int_0^\infty \left(e^{(-t)} t^{(x-1)} \right) dt.$$

Therefore, whenever the integral defined by real x converges, so will the integral defined by complex z. As proved, this occurs whenever $x > 0$, so it occurs whenever $z \in \mathcal{R}(0, \infty)$. $\square$

This integral is the crux of the gamma function, though before the true gamma function is defined it may be used to define (in what is admittedly not standard practice) what this article shall call the pre-gamma function. Typically (see [3], [4], [5], etc.) this "pre-gamma" function is simply called the gamma function, but for the sake of not having to redefine what exactly the gamma function is post-continuation, this name will be used.

Definition 1.2 (Pre-gamma function). Let $\Gamma_0 : \mathcal{R}(0, \infty) \to \mathbb{C}$ be defined by

$$\Gamma_0(z) = \int_0^\infty \left(e^{(-t)} t^{(z-1)} \right) dt. \tag{1.2}$$

Then, Γ_0 is the **pre-gamma function**. [3]

A cursory plot of the pre-gamma function on the real axis (Figure 1.1) seems to indicate that as $z \to 0$ from the positive real axis, $|\Gamma_0(z)| \to \infty$. However, another cursory plot on the ray one unit

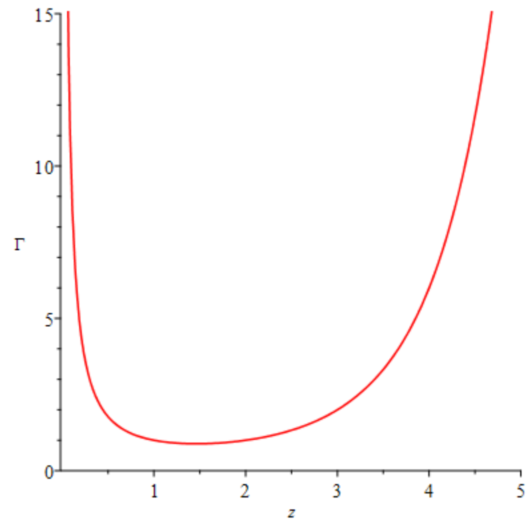

Figure 1.1: $\Gamma_0(z)$ for $z \in (0, 5]$

above the positive real axis (as in, $\{x+i \mid x > 0\}$) (Figure 1.2) shows no such tendency in the case that $z \to i$ along the ray.

By Lemma 1.1, the integral definition of the pre-gamma function is well-defined, as the integral defining $\Gamma_0(z)$ converges for all $z \in \mathcal{R}(0, \infty)$. However, this only defines a function with a domain consisting of a half-plane. Thus, an attempt is made to use analytic continuation to find a larger domain for this function. The figures seem to indicate that any well-defined analytic continuation of $\Gamma_0(z)$ will have a nonremovable singularity when $z = 0$, but that there need not be a singularity at, for example, $z = 1 + i$. This analysis will turn out to be true.

That Γ_0 is analytic at all is something that, unfortunately, will not

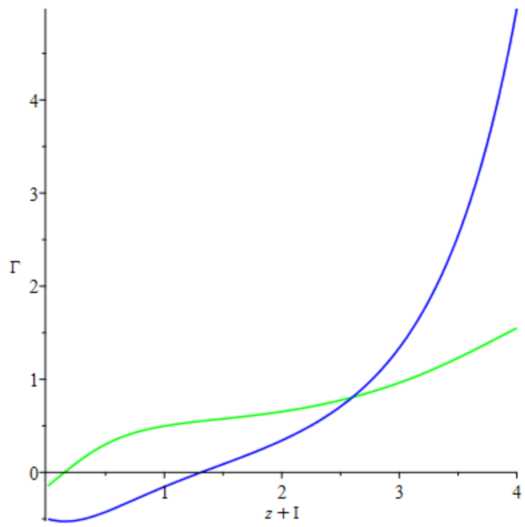

Figure 1.2: $\Gamma_0(z)$ for $z = x + i$, where $x \in (0, 4]$, $\mathrm{Re}(\Gamma_0(z))$ is given in green, $\mathrm{Im}(\Gamma_0(z))$ in blue

be proved in this article, though a proof is outlined in [3] for instance , utilizing what is known as Morera's theorem. However, if it is assumed that Γ_0 is analytic, analytic continuation may be used to get a larger domain for the function by first proving a relation.

Lemma 1.2. *For all* $z \in \mathcal{R}(0, \infty)$*,*

$$\Gamma_0(z) = \frac{\Gamma_0(z+1)}{z}. \tag{1.3}$$

Proof. Using integration by parts on (1.2), it can then be seen that for all $z \in \mathcal{R}(0,\infty)$,

$$\Gamma_0(z) = \left(e^{(-t)}\left(\frac{t^z}{z}\right)\right|_{t=0}^{t=\infty} - \int_0^\infty \left(-e^{(-t)}\left(\frac{t^z}{z}\right)\right) dt.$$

The boundary term is equal to 0 (as both limits are zero), so this reduces to

$$\Gamma_0(z) = \frac{\int_0^\infty \left(e^{(-t)}t^z\right) dt}{z} = \frac{\Gamma_0(z+1)}{z}. \text{ [3]}$$

□

Now that this relation has been found for the pre-gamma function's arguments, it is reasonable to conclude that the relation would continue to be true for the "actual" gamma function's arguments, as well. As an example, consider $\Gamma_1 : \mathcal{R}(-1,\infty)\backslash\{0\} \to \mathbb{C}$ defined by

$$\Gamma_1(z) = \begin{cases} \Gamma_0(z) & , z \in \mathcal{R}(0,\infty) \\ \dfrac{\Gamma_0(z+1)}{z} & , z \in \mathcal{R}(-1,0]. \end{cases}$$

This function holds the property that the pre-gamma function holds, in that

$$\Gamma_1(z) = \frac{\Gamma_1(z+1)}{z},$$

as anywhere within the domain of Γ_0, it can be seen that both $\Gamma_1(z) = \Gamma_0(z)$ and $\Gamma_1(z+1) = \Gamma_0(z+1)$, and outside of the domain of

Γ_0 but within the domain of Γ_1, it is *still* true that $\Gamma_1(z+1) = \Gamma_0(z+1)$, so the definition of Γ_1 reduces to the aforementioned relation there.

Thus, Γ_1 is a function that holds the same property that Γ_0 does. Unfortunately, the domain of Γ_1 does not include 0, as the relation would turn into a division by zero. This is not a removable singularity and, as will be seen, corresponds to a pole.

Now, moving from Γ_0 to Γ_1 expanded the domain of the pre-gamma function from $\mathcal{R}(0, \infty)$ to $\mathcal{R}(-1, \infty)\backslash\{0\}$, so the domain of the pre-gamma function has been shifted to the left by one (with an isolated singularity added to the mix). There is no reason to simply stop, however, as the domain can be shifted to the left by one once again by applying the recursive definition again to all elements of $\mathcal{R}(-2, 1]\backslash\{1\}$, where once again there will be a new pole (this time at 1), as otherwise the recursive definition would require knowing the value of $\Gamma_1(z)$ at $z = 0$, where it is not defined.

Thus, using the recursive relation repeatedly to continue shifting the domain of the pre-gamma function, the final result is as follows.

Definition 1.3 (Gamma function)**.** Let $\Gamma : \mathbb{C}\backslash\mathbb{N}^- \to \mathbb{C}$ be defined by

$$\Gamma(z) = \begin{cases} \Gamma_0(z) & , z \in \mathcal{R}(0, \infty) \\ \\ \dfrac{\Gamma_0(z+n)}{\prod_{j=0}^{n-1}(z+j)} & , \exists n \in \mathbb{Z}^+ \text{ such that } z \in \mathcal{R}(-n, -n+1]. \end{cases} \tag{1.4}$$

Then, Γ is called the **gamma function**. [5]

It is worth noting that this is, in a sense, a nonstandard definition (or, at least, a different definition than that given by all the sources) for the "full" gamma function, however [5] makes a point of noting that the relation used to derive this holds for their definition of the gamma function and, given the same definition of Γ_0, this is the unique analytic continuation that holds that property, so this definition must be equivalent.

The gamma function shows up occasionally in a few different areas. One such area in in probability, in what is known as the gamma distribution. [2]

Once again, that Γ is an analytic function will have to be assumed; however, that it happens to (by definition and recursion) have the exact same relationship that Γ_0 has, namely:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \tag{1.5}$$

shows that it must be intimately related to the pre-gamma function, so to claim that this must be an analytic continuation of the other (note that if it is analytic it fits the bill for an analytic continuation) is not too far-fetched.

1.3 Properties of the Gamma Function

There are multiple properties of the gamma function that are worth mentioning, some of them useful in examining the Riemann zeta function (believe it or not) and others as simple curiosities. They will be examined in this section.

Lemma 1.3. *For all* $z \in \mathbb{C}\backslash\mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \text{ [3]} \tag{1.6}$$

The proof of this relation is beyond the scope of this paper, though it is outlined in [3] as an exercise; it involves using what the source refers to as the beta function. This relation, however, can be used to prove a few other properties of the gamma function. For example, in the case that $z = \frac{1}{2}$,

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)}, \text{ so } \Gamma^2\left(\frac{1}{2}\right) = \pi.$$

Thus,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \text{ [3]} \tag{1.7}$$

It is, of course, possible to compute values of $\Gamma(z)$ directly from the integral equation definition. For example,

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty \left(e^{(-t)} t^{(1-1)}\right) dt \\
&= \int_0^\infty \left(e^{(-t)}\right) dt \\
&= -\left(e^{(-t)}\right|_0^\infty \\
&= 1.
\end{aligned}$$

Thus, writing it in a seemingly odd way,

$$\Gamma(1) = 0!. \tag{1.8}$$

In fact, there is a deep connection between the gamma function and the factorial function.

Theorem 1.1. *For all* $n \in \mathbb{Z}^+$,

$$\Gamma(n) = (n-1)!. \tag{1.9}$$

Proof. By (1.8), $\Gamma(1) = 0!$, so $\Gamma(1) = (1-1)!$.

Suppose that $j \in \mathbb{Z}^+\backslash\{1\}$ and that $\Gamma(j) = (j-1)!$. By (1.5),

$$\Gamma(j) = \frac{\Gamma(j+1)}{j}, \text{ so}$$

$$\Gamma(j+1) = j\Gamma(j) = j(j-1)! = j!.$$

Thus, by induction, for all $n \in \mathbb{Z}^+$,

$$\Gamma(n) = (n-1)!. \text{ [4]}$$

□

Combining Lemma 1.3 and Theorem 1.1, something about the gamma function that will be important when examining the Riemann zeta function comes to light.

Theorem 1.2. *For all* $z \in \mathbb{C}\backslash\mathbb{N}^-$,

$$\Gamma(z) \neq 0. \tag{1.10}$$

Proof. Suppose $z \in \mathbb{Z}^+$. Then, by Theorem 1.1,

$$\Gamma(z) = (z-1)! \neq 0.$$

Suppose, instead, that $z \in \mathbb{C} \backslash \mathbb{Z}$. Then, by Lemma 1.3,

$$\Gamma(z) = \frac{\pi}{\Gamma(1-z)\sin(\pi z)} \neq 0.$$

Thus, the gamma function is nowhere zero, as those two sets encompass the entirety of the gamma function's domain. [5] $\square$

Another fact about the gamma function, which seems like a strange thing to note, will also come in handy when examining the Riemann zeta function. It is actually used in [8] when deriving the analytic continuation of the Riemann zeta function, though it is used without proof or explicitly stating that it is being used.

Lemma 1.4. *For all* $z \in \mathcal{R}(-1, 0)$*,*

$$\Gamma(-z)\sin\left(\frac{\pi z}{2}\right) = -\int_0^\infty \left(t^{(-1-z)}\sin(t)\right) dt. \tag{1.11}$$

This, too, shall be, unfortunately, given without proof.

Finally, as one last curiosity, one may determine the general character of Γ over the complex plane.

Theorem 1.3. Γ *is meromorphic on* $\mathbb{C}$*, with poles at each nonpositive integer of order* 1*. Furthermore, if* $R : \mathbb{N} \to \mathbb{C}$ *is defined such that* $R(n)$

is the residue of the pole of $\Gamma(z)$ *at* $z = -n$*, then*

$$R(n) = \frac{(-1)^n}{n!}. \tag{1.12}$$

Proof. Let $n \in \mathbb{N}$. When z is near $-n$,

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)(z+n)}.$$

With that in mind, let $H : (\mathbb{C}\backslash\mathbb{N}^{-}) \cup \{-n\} \to \mathbb{C}$ be defined by

$$H(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)}.$$

Thus, everywhere but at $z = -n$, $H(z) = \Gamma(z)(z+n)$. It can be seen that attempting to let $z = -n$ provides a perfectly working value for $H(z)$, so the pole at $-n$ must be of order 1. Furthermore, it will have residue of

$$R(n) = H(-n) = \frac{\Gamma(1)}{-n(1-n)\cdots(-1)} = \frac{(-1)^n}{n(n-1)\cdots(1)},$$

so

$$R(n) = \frac{(-1)^n}{n!}. \text{ [4]}$$

□

At first, the gamma function may seem completely unrelated to the Riemann zeta function (assuming prior familiarity with the zeta function). However, an intimate knowledge of the gamma function is actually quite helpful to understanding how the zeta function works. Figures 1.3 and 1.4 show various properties of the full gamma function on the complex plane.

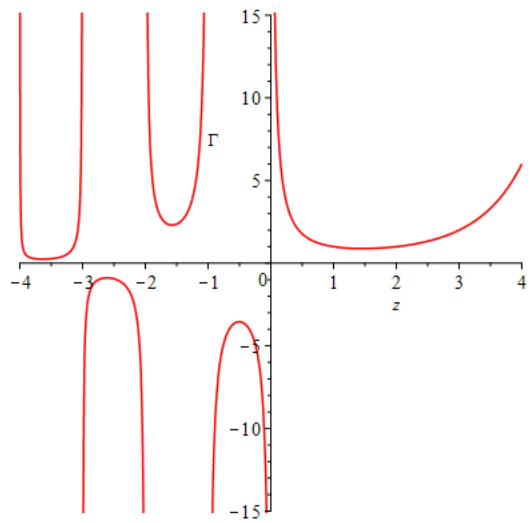

Figure 1.3: $\Gamma(z)$ for $z \in (-4, 4]\backslash\{-3, -2, -1, 0\}$

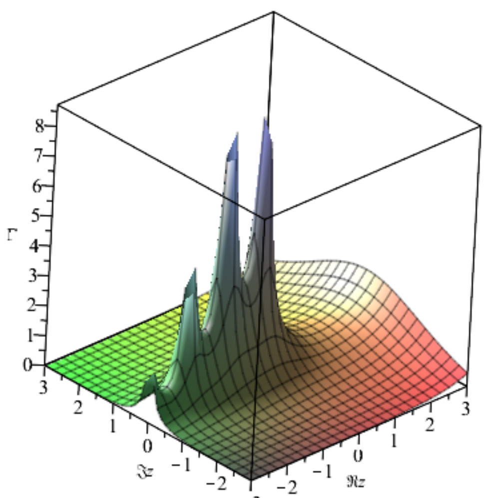

Figure 1.4: $|\Gamma(z)|$ (cut off in height for the poles) for $z = x + iy$, where $(x, y) \in ([-3, 3] \times [-3, 3]) \setminus \{(-3, 0), (-2, 0), (-1, 0), (0, 0)\}$

Chapter 2

The Riemann Zeta Function

2.1 The First Continuation

The derivation of the gamma function was a particularly useful example of analytic continuation, as the gamma function will prove useful in this chapter. For now, it is time to consider a partiular sum, which shall form the base upon which the structure of the Riemann zeta function will be built.

Lemma 2.1. *For all $s \in \mathcal{R}(1, \infty)$,*

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^s}\right) \textit{ converges.}$$

Proof. Suppose $s > 1$. Then, by the integral test,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right) \text{ converges if } \int_1^{\infty}\left(\frac{1}{x^s}\right)dx \text{ converges.}$$

As the integral converges whenever $s > 1$, so does the sum.

Now suppose, more generally, that $s \in \mathcal{R}(1, \infty)$. Let $\sigma \in \mathbb{R}$ and let $t \in \mathbb{R}$, where $s = \sigma + it$. Then, noting that

$$|n^s| = \left|e^{(s\ln(n))}\right| = \left|e^{((\sigma+it)\ln(n))}\right| = \left|e^{(\sigma\ln(n))}\right| = |n^{\sigma}|,$$

it can be seen that

$$\left|\sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right)\right| \leq \left|\sum_{n=1}^{\infty}\left(\frac{1}{n^{\sigma}}\right)\right|,$$

so the sum determined by complex s converges whenever the sum determined by real σ converges, which occurs whenever real $\sigma > 1$, or when $s \in \mathcal{R}(1, \infty)$. $\square$

In the previous parts of this article, z was used as a sort of general complex variable (with real part x and imaginary part y). In this section, when referring to the arguments of the Riemann zeta function, the complex variable is denoted by s, with real part σ and imaginary part t.

As with the gamma function, this sum may be used to define a function that this article will call the pre-zeta function.

Definition 2.1 (Pre-zeta function). Let $\zeta_0 : \mathcal{R}(1, \infty) \to \mathbb{C}$ be defined such that

$$\zeta_0(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} \right). \tag{2.1}$$

Then, ζ_0 is the **pre-zeta function**.

Figures 2.1 and 2.2, indicating plots of the pre-zeta function, show similar behaviour to how the gamma function behaved previously. In the case of the gamma function, it was an approach to the border of $\mathcal{R}(0, \infty)$. In the case of the pre-zeta function, it is an approach to the border of $\mathcal{R}(1, \infty)$. Thus, a similar expectation of the ability to analytically continue the function (with a pole at 1) exists. After analytically continuing it, the properties of this new "full" zeta function may be studied.

As with the pre-gamma function, the pre-zeta function is more of a placeholder definition for what will be the true Riemann zeta function. In order to analytically continue it past the half-plane of definition, the first method mentioned in chapter two of [8] will be used. In order to use this, it is important to consider the following summation rule.

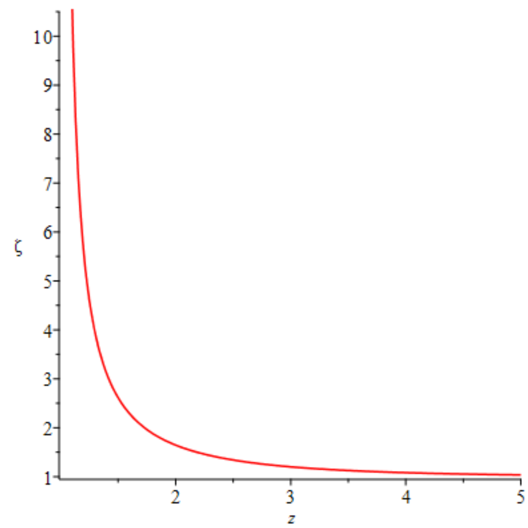

Figure 2.1: $\zeta_0(z)$ for $z \in (1, 5]$

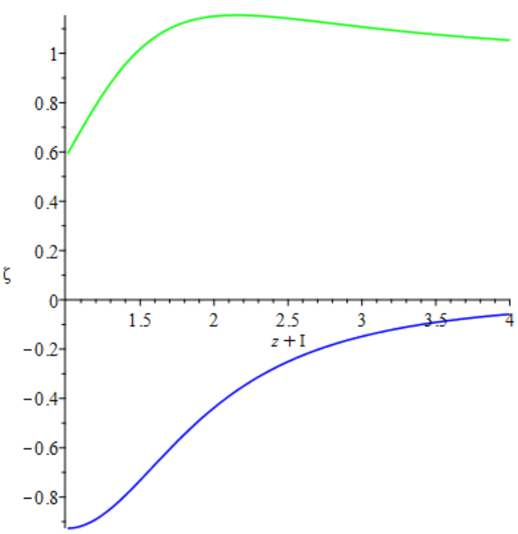

Figure 2.2: $\zeta_0(z)$ for $z = x + i$, where $x \in (1, 4]$, $\mathrm{Re}(\zeta_0(z))$ is given in green, $\mathrm{Im}(\zeta_0(z))$ in blue

Lemma 2.2. *Let* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}$ *such that* $\lfloor b \rfloor > \lfloor a \rfloor$*. Let* $\phi : [\lfloor a \rfloor, b] \to \mathbb{C}$ *be continuously differentiable on* $(\lfloor a \rfloor, b)$*. Then,*

$$\begin{aligned}\sum_{n=\lfloor a \rfloor+1}^{\lfloor b \rfloor} (\phi(n)) = \int_a^b (\phi(x))\,dx + \int_a^b \left(\left(x - \lfloor x \rfloor - \frac{1}{2}\right)\phi'(x)\right)dx \\ + \left(a - \lfloor a \rfloor - \frac{1}{2}\right)\phi(a) - \left(b - \lfloor b \rfloor - \frac{1}{2}\right)\phi(b).\end{aligned} \tag{2.2}$$

The skeleton of the proof of this rule is outlined in [8], though it is tedious and not particularly interesting. However, using this summation rule, for any $s \in \mathcal{R}(1,\infty)$ and $b \in \mathbb{Z}^+ \backslash \{1\}$,

$$\begin{aligned}\sum_{n=\lfloor 1 \rfloor+1}^{\lfloor b \rfloor} \left(\frac{1}{n^s}\right) = \int_1^b \left(\frac{1}{x^s}\right)dx - \int_1^b \left(\left(x - \lfloor x \rfloor - \frac{1}{2}\right)\frac{s}{x^{(s+1)}}\right)dx \\ + \left(1 - \lfloor 1 \rfloor - \frac{1}{2}\right)\frac{1}{1^s} - \left(b - \lfloor b \rfloor - \frac{1}{2}\right)\frac{1}{b^s}, \text{ so}\end{aligned}$$

$$\sum_{n=2}^{b} \left(\frac{1}{n^s}\right) = \int_1^b \left(\frac{1}{x^s}\right)dx - s\int_1^b \left(\frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{(s+1)}}\right)dx - \frac{1}{2} + \frac{1}{2b^s}.$$

In the limit that $b \to \infty$, this becomes

$$\begin{aligned}\sum_{n=2}^{\infty} \left(\frac{1}{n^s}\right) &= \left(\frac{1}{x^{(1-s)}(1-s)}\right|_1^\infty - s\int_1^\infty \left(\frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{(s+1)}}\right)dx - \frac{1}{2}; \\ \zeta_0(s) - 1 &= s\int_1^\infty \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right)dx + \frac{1}{s-1} - \frac{1}{2}.\end{aligned}$$

Thus,

$$\zeta_0(s) = s\int_1^\infty \left(\frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right) dx + \frac{1}{s-1} + \frac{1}{2}. \tag{2.3}$$

It is worth noting that, for all $x \in \mathbb{R}$, $\left|\lfloor x\rfloor - x + \frac{1}{2}\right| \leq \frac{1}{2}$, so the integral in (2.3) converges whenever $s \in \mathcal{R}(0,\infty)$. Also worth noting is that the second term in (2.3) is defined whenever $s \neq 1$. Thus, the righthand side of (2.3) is defined whenever $s \in \mathcal{R}(0,\infty)\backslash\{1\}$. Thus, an analytic continuation of the pre-zeta function may be formed.

Definition 2.2 (First analytic continuation of the pre-zeta function). Let $\zeta_1 : \mathcal{R}(0,\infty)\backslash\{1\} \to \mathbb{C}$ be defined such that

$$\zeta_1(s) = s\int_1^\infty \left(\frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right) dx + \frac{1}{s-1} + \frac{1}{2}. \tag{2.4}$$

Then, ζ_1 is the **first analytic continuation of the pre-zeta function**.

That ζ_1 is analytic is stated by [8] and will be assumed going forwards. Interestingly enough, this definition allows one to calculate the value of $\zeta_1(s)$ for $s \in \mathcal{R}(0,1]\backslash\{1\}$ without first knowing the value of the zeta function anywhere else. That is to say, this analytic continuation definition is not recursive, unlike the definition given for the gamma function.

2.2 Further Continuation

To continue the zeta function further, it is useful to note the following thing.

Lemma 2.3. *For all* $k \in \mathbb{Z}$*,*

$$\int_k^{k+1} \left(\lfloor x \rfloor - x + \frac{1}{2}\right) dx = 0. \tag{2.5}$$

Proof. Note that for all $x \in (k, k+1)$, $\lfloor x \rfloor = k$. Thus,

$$\begin{aligned}
\int_k^{k+1} \left(\lfloor x \rfloor - x + \frac{1}{2}\right) dx &= \int_k^{k+1} \left(k - x + \frac{1}{2}\right) dx \\
&= \left(kx - \frac{x^2}{2} + \frac{x}{2}\right|_k^{k+1} \\
&= k^2 + k - \frac{k^2}{2} - k - \frac{1}{2} + \frac{k}{2} + \frac{1}{2} - k^2 + \frac{k^2}{2} - \frac{k}{2} \\
&= 0. \text{ [8]}
\end{aligned}$$

□

Thus, there exists $M > 0$ such that for all $a \in \mathbb{R}$ and $b > a$,

$$\left| \int_a^b \left(\lfloor x \rfloor - x + \frac{1}{2}\right) dx \right| < M. \tag{2.6}$$

This will be useful in proving the next lemma.

Lemma 2.4. *For all* $s \in \mathcal{R}(-1, \infty)$,

$$\int_1^\infty \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}} \right) dx \textit{ converges.}$$

Proof. Using integration by parts,

$$\begin{aligned} \int_1^\infty \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}} \right) dx = &\left(\frac{\int_1^x \left(\lfloor y \rfloor - y + \frac{1}{2} \right) dy}{x^{(s+1)}} \right|_1^\infty \\ &+ (s+1) \int_1^\infty \left(\frac{\int_1^x \left(\lfloor y \rfloor - y + \frac{1}{2} \right) dy}{x^{(s+2)}} \right) dx. \end{aligned} \tag{2.7}$$

The boundary term works out to be 0 as both limits are 0 (remember that (2.6) means that the numerator of the boundary term is always bunded, so as $x \to \infty$ the numerator remains bounded but the denominator approaches infinity), and the integral in the second term in the righthand side converges whenever $s \in \mathcal{R}(-1, \infty)$ (using Lemma 0.1 and (2.6)), so the original integral converges whenever $s \in \mathcal{R}(-1, \infty)$. $\square$

Thus, despite not really having done much, a second analytic continuation of the pre-zeta function may be defined.

Definition 2.3 (Second analytic continuation of the pre-zeta function). Let $\zeta_2 : \mathcal{R}(-1, \infty) \backslash \{1\} \to \mathbb{C}$ be defined such that

$$\zeta_2(s) = s \int_1^\infty \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}} \right) dx + \frac{1}{s-1} + \frac{1}{2}. \tag{2.8}$$

Then, ζ_2 is the **second analytic continuation of the pre-zeta function**.

In order to continue the zeta function to the rest of the complex plane, it would be nice to be able to write ζ_2 as a single integral without the other two terms in the righthand side of (2.8). In fact, this is possible.

Lemma 2.5. *For all* $s \in \mathcal{R}(-1, 0)$*,*

$$s\int_0^1 \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right) dx = \frac{1}{s-1} + \frac{1}{2}. \tag{2.9}$$

Proof. For all $x \in (0, 1)$, $\lfloor x \rfloor = 0$, so

$$\begin{aligned} s\int_0^1 \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right) dx &= s\int_0^1 \left(-\frac{1}{x^s} + \frac{1}{2x^{(s+1)}}\right) dx \\ &= s\left(\frac{1}{(s-1)x^{(s-1)}} - \frac{1}{2sx^s}\right|_0^1 \\ &= \frac{s}{s-1} - \frac{1}{2} = \frac{s-s+1}{s-1} + \frac{1}{2}. \end{aligned}$$

Thus,

$$s\int_0^1 \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right) dx = \frac{1}{s-1} + \frac{1}{2}. \text{ [8]}$$

□

Thus, for all $s \in \mathcal{R}(-1, 0)$,

$$\zeta_2(s) = s \int_0^{\infty} \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}} \right) dx. \tag{2.10}$$

At this point, it is useful to note that $\lfloor x \rfloor - x + \frac{1}{2}$ is a periodic function with period 1, and thus has a Fourier series anywhere except its discontinuities (the integers) of

$$\sum_{n=0}^{\infty} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \right),$$

where

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(\lfloor x \rfloor - x + \frac{1}{2} \right) \cos(2\pi n x) \right) dx, \tag{2.11}$$

and

$$b_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(\lfloor x \rfloor - x + \frac{1}{2} \right) \sin(2\pi n x) \right) dx. \text{ [7]} \tag{2.12}$$

Now, if $y > 0$, then $\lfloor -y \rfloor + y + \frac{1}{2} = -\lfloor y \rfloor - 1 + y + \frac{1}{2} = -\lfloor y \rfloor + y - \frac{1}{2}$, so $\lfloor x \rfloor - x + \frac{1}{2}$ is an odd function, so, as $\cos(2\pi n x)$ is an even function, $a_n = 0$ for all n. Note that $b_0 = 0$, so the Fourier series of this function becomes

$$\sum_{n=1}^{\infty} \left(b_n \sin(2\pi n x) \right),$$

where, as $\sin(2\pi nx)$ is an odd function,

$$b_n = 4\int_0^{\frac{1}{2}} \left(\left(-x+\frac{1}{2}\right)\sin(2\pi nx)\right)dx. \tag{2.13}$$

Using integration by parts, this becomes

$$\begin{aligned} b_n &= -\frac{4}{2\pi n}\left(\left(-x+\frac{1}{2}\right)\cos(2\pi nx)\right|_0^{\frac{1}{2}} + 4\int_0^{\frac{1}{2}}(\cos(2\pi nx))\,dx \\ &= \frac{1}{\pi n} + \frac{4\sin(\pi n)}{2\pi n} \\ &= \frac{1}{\pi n}. \end{aligned}$$

Thus, for all $x \in \mathbb{R}\backslash\mathbb{Z}$,

$$\lfloor x \rfloor - x + \frac{1}{2} = \sum_{n=1}^{\infty}\left(\frac{\sin(2\pi nx)}{\pi n}\right). \text{ [8]} \tag{2.14}$$

Thus, for all $s \in \mathcal{R}(-1,0)$, using $y = 2\pi nx$ substitution and then Lemma 1.4 and then (1.5), and then noting that, for all $s \in \mathcal{R}(-1,0)$,

$\zeta_0(1-s) = \zeta_2(1-s)$,

$$\begin{aligned}
\zeta_2(s) &= \frac{s}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^{\infty}\left(\frac{\sin(2\pi nx)}{x^{(s+1)}}\right)dx\right) \\
&= \frac{s}{\pi}\sum_{n=1}^{\infty}\left(\frac{(2\pi n)^s}{n}\int_0^{\infty}\left(\frac{\sin(y)}{y^{(s+1)}}\right)dy\right) \\
&= 2^s\pi^{(s-1)}s\sum_{n=1}^{\infty}\left(\frac{1}{n^{(1-s)}}\right)\int_0^{\infty}\left(\frac{\sin(y)}{y^{(s+1)}}\right)dy \\
&= -2^s\pi^{(s-1)}s\zeta_0(1-s)\Gamma(-s)\sin\left(\frac{\pi s}{2}\right) \\
&= 2^s\pi^{(s-1)}\zeta_2(1-s)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right). \text{ [8]}
\end{aligned}$$

In conclusion, for all $s \in \mathcal{R}(-1,0)$,

$$\zeta_2(s) = 2^s\pi^{(s-1)}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta_2(1-s). \tag{2.15}$$

Thus, a relation between two different values of the zeta function has been derived. The righthand side of (2.15) is actually defined whenever $s \in \mathcal{R}(-\infty,-1]$ (it is defined for more values but those are the only ones needed), and so may be used to continue the zeta function to the rest of the complex plane.

Definition 2.4 (Riemann zeta function)**.** Let $\zeta : \mathbb{C}\backslash\{1\} \to \mathbb{C}$ be defined by

$$\zeta(s) = \begin{cases} \zeta_2(s) & , s \in \mathcal{R}[0, \infty) \\ \\ 2^s \pi^{(s-1)} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta_2(1-s) & , s \in \mathcal{R}(-\infty, 0). \end{cases} \tag{2.16}$$

Then, ζ is called the **Riemann zeta function**. [5]

The Riemann zeta function is a very important function, with entire books having been written about it (such as [8]). To close out this article, a few properties of it will be listed.

2.3 Properties of the Riemann Zeta Function

For most infinite sums, the only information that can be derived analytically from them is whether they converge or diverge. For some classes of convergent infinite sums, however, there are well-known methods to calculate their values. Consider the case of finding the value of p when

$$p = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right).$$

Note that this is equivalent to finding the value of $\zeta(2)$.

In order to determine this, the fifth proof method in [1] will be used, which starts off with something that will remain unproven: for all $x \in [0,1]$,

$$x(1-x) = \frac{1}{6} - \sum_{n=1}^{\infty} \left(\frac{\cos(2\pi n x)}{\pi^2 n^2} \right). \text{ [1]} \tag{2.17}$$

In particular, when $x = 0$,

$$\sum_{n=1}^{\infty} \left(\frac{1}{\pi^2 n^2} \right) = \frac{1}{6},$$

so it comes to light that

$$\zeta(2) = \frac{\pi^2}{6}. \tag{2.18}$$

That was an example of finding a value of ζ using the sum definition. Consider, now, the integral definition, which is to say, that when $s \in \mathcal{R}[0,\infty)\backslash\{1\}$ that

$$\zeta(s) = s \int_1^{\infty} \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}} \right) dx + \frac{1}{s-1} + \frac{1}{2}.$$

Well, when $s = 0$, this turns into $-\frac{1}{2}$ quite easily, so

$$\zeta(0) = -\frac{1}{2}. \tag{2.19}$$

It is also possible to calculate values of ζ using the recursive definition. For example,

$$\zeta(-1) = -\frac{\Gamma(2)\zeta(2)}{2\pi^2}, \text{ so}$$

$$\zeta(-1) = -\frac{1}{12}. \tag{2.20}$$

One thing worth examining about the Riemann zeta function is where its zeros are.

Theorem 2.1. *For all* $n \in \mathbb{Z}^+$*,*

$$\zeta(-2n) = 0. \tag{2.21}$$

Proof. For all $n \in \mathbb{Z}^+$, $-2n \in \mathcal{R}(-\infty, 0)$, so

$$\zeta(-2n) = 2^{(-2n)}\pi^{(-2n-1)}\sin(-\pi n)\Gamma(2n+1)\zeta(2n+1) = 0. \text{ [8]}$$

$\square$

It will be stated without proof (though a proof is given in [8]), but it is true that the sum definition of the Riemann zeta function may be rewritten as an infinite product.

Lemma 2.6. *Let $\mathcal{P}$ be the set of all primes. For all $z \in \mathcal{R}(1, \infty)$,*

$$\zeta(s) = \prod_{p \in \mathcal{P}} \left(\frac{1}{1 - p^{(-s)}} \right). \ [8] \tag{2.22}$$

Note that this product can never be zero, as none of the terms can be zero and as $s \to \infty$ the factors approach 1. Thus, the Riemann zeta function can never be zero in the region $\mathcal{R}(1, \infty)$. Examining the functional definition, it is never zero except for the "trivial" zeros at the negative even integers, as none of the factors is ever zero, due to simple exponent rules, how the sine function behaves, and Theorem 1.2 and the fact the zeta function is never zero on $\mathcal{R}(1, \infty)$. Thus, the Riemann zeta function is never zero anywhere, other than the trivial zeros or in the region $\mathcal{R}[0, 1]$.

In fact, there exists a conjecture, known as the Riemann zeta hypothesis, which claims that all nontrivial zeros of the Riemann zeta function have real part of $\frac{1}{2}$. [8]

To finish off this article, the character of the Riemann zeta function will be examined.

Theorem 2.2. *ζ is meromorphic on $\mathbb{C}$, with a pole at 1 of order 1. Furthermore, if R is the residue of the pole of $\zeta(z)$ at $z = 1$, then*

$$R = 1. \tag{2.23}$$

Proof. If one assumes that the functional relation defining the Riemann zeta function actually works on the positive-reals half-plane as well as the negative-reals half-plane (ignoring removable singularities), then near $s = 1$,

$$\begin{aligned}\zeta(s) &= 2^s \pi^{(s-1)} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \\ &= \left(\frac{1}{1-s}\right)\left(2^s \pi^{(s-1)} \sin\left(\frac{\pi s}{2}\right) \Gamma(2-s)\zeta(1-s)\right).\end{aligned}$$

With that in mind, let $H : \mathbb{C} \to \mathbb{C}$ be defined by

$$H(z) = -2^s \pi^{(s-1)} \sin\left(\frac{\pi s}{2}\right) \Gamma(2-s)\zeta(1-s).$$

Then, noting that $H(z)$ is perfecly defined at $z = 1$, it becomes clear that the pole is of order 1 and that the residue is

$$R = H(1) = -2\Gamma(1)\zeta(0) = 1.$$

Alternatively, one may use (2.8) to note that near $s = 1$,

$$\zeta(s) = s \int_1^\infty \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right) dx + \frac{1}{s-1} + \frac{1}{2}.$$

With that in mind, let $H : \mathbb{C} \to \mathbb{C}$ be defined by

$$H(s) = s(s-1) \int_1^\infty \left(\frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{(s+1)}}\right) dx + 1 + \frac{s-1}{2}.$$

Here, it is quite easy to see that $H(1) = 1$, which also completes the proof. $\square$

With that, the analysis of the gamma and Riemann zeta functions comes to an end. Figures 2.3, 2.4, and 2.5 show off how the full Riemann zeta function behaves on the complex plane.

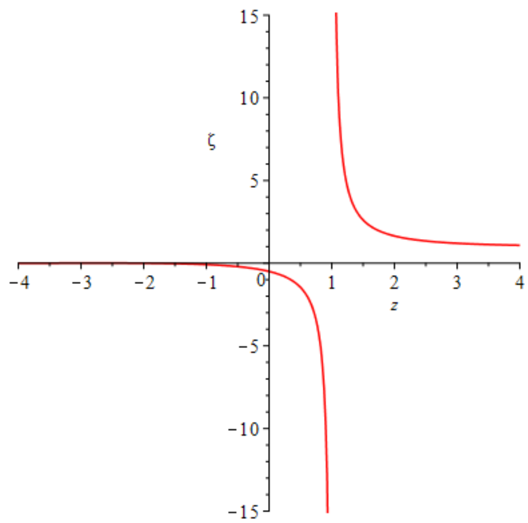

Figure 2.3: $\zeta(z)$ for $z \in [-4, 4]\backslash\{1\}$

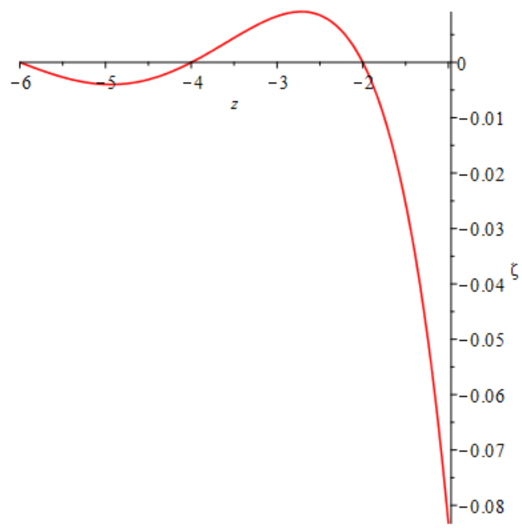

Figure 2.4: $\zeta(z)$ for $z \in [-6, -1]$, better showcasing the trivial zeros

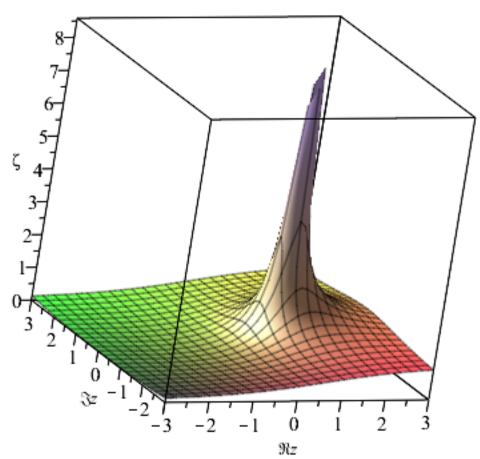

Figure 2.5: $|\zeta(z)|$ (cut off in height for the pole) for $z = x+iy$, where $(x, y) \in ([-3, 3] \times [-3, 3]) \setminus \{(1, 0)\}$

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